Solutions to Gelfand problems

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$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = \lambda f(u)\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

M, Gelfand type problem for singular quadratic quasilinear equations, *Nonlinear Differential Equations and Applications NoDEA* (2016).

3. 1-homogeneous p-laplacian

$$\begin{cases} -\Delta_p^N u = \lambda f(u)\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

Work in preparation with J. Carmona and J.D. Rossi.

Existence of positive solutions

$$\begin{cases} -\Delta u = \lambda \, e^u \, , & \text{in } \Omega, \\ u = 0 \, , & \text{on } \partial\Omega, \end{cases} \tag{1}$$

- Model for the thermal reaction process in a combustible.
- I.M. Gelfand, Some problems in the theory of quasilinear equations, *Amer. Math. Soc. Transl.* (1963).

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- Model for the thermal reaction process in a combustible.
- I.M. Gelfand, Some problems in the theory of quasilinear equations, *Amer. Math. Soc. Transl.* (1963).
- No solutions for λ large. Even in the weak sense.
- Leray-Schauder continuation methods. Existence of an unbounded continuum of solutions.

$\Omega = B_1(0)$: Gidas-Ni-Niremberg $\rightarrow$ solutions are radially, symmetric and satisfy

$$\begin{cases} -u'' - \frac{N-1}{r} u' = \lambda e^u, & r \in [0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{2}$$

where $u(r) := u(|x|)$.

- $1 \leq N \leq 2$: There exists $\lambda^* > 0$ such that (2) has exactly one solution for $\lambda = \lambda^*$ and exactly two solutions for each $\lambda \in (0, \lambda^*)$.
- $2 < N < 10$: Eq. (2) has a continuum of solutions which oscillates around the line $\lambda = 2(N-2)$; with the amplitude of oscillations tending to zero, as $u(0) = \|u\|_\infty \rightarrow \infty$.
- $N \geq 10$: Eq. (2) has a unique solution for each $\lambda \in (0, 2(N-2))$ and no solutions for $\lambda \geq 2(N-2)$: Moreover, $\|u\|_\infty \rightarrow \infty$ as $\lambda \rightarrow 2(N-2)$.

Classical Gelfand type problems

$$\begin{cases} -\Delta u = \lambda f(u)\,, & \text{in } \Omega, \\ u > 0\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases} \qquad (G_\lambda)$$

$\Omega \subset \mathbb{R}^N$ bounded, smooth, $\lambda > 0$, $f(0) > 0$, derivable, increasing, convex and superlinear $\left(\lim_{s\to\infty} \dfrac{f(s)}{s} = \infty \right)$.

$(f(u) = e^u$, $(1+u)^p$, $\frac{1}{(1-u)^k}$ ... with $p > 1$, $k > 0)$.

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Crandall-Rabinowitz (1973)

There exists a positive number λ^* called the *extremal parameter* such that

- If $\lambda < \lambda^*$ the problem (G_λ) admits a minimal bounded solution w_λ.
- If $\lambda > \lambda^*$ the problem (G_λ) admits no solution.

Even more, the sequence $\{w_\lambda\}$ is increasing.

What happens when $\lambda = \lambda^*$?

Let $u^*(x) := \lim_{\lambda \to \lambda^*} w_\lambda(x)$

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Crandall-Rabinowitz (1975) $f(u) = (1+u)^p, \; p > 1$

The *extremal solution* u^* is bounded in dimensions

$$N < 4 + 2\frac{p}{p-1} + 4\sqrt{\frac{p}{p-1}}$$

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Mignot-Puel (1980) $f(u) = e^u$

- The *extremal solution* u^* is bounded in dimensions $N \leq 9$ for any domain Ω.
- When $N \geq 10$, $u^*(x) = \log \frac{1}{|x|^2}$ is the singular extremal solution for $\Omega = B_1(0)$.

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- u^* is bounded for $N = 4$, S. Villegas (2013).
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The stability condition, i.e., u_λ satisfies

$$\int_\Omega |\nabla \phi|^2 \geq \lambda \int_\Omega f'(u_\lambda)\phi^2, \quad \forall\, \phi \in \mathcal{C}_c^\infty(\Omega),$$

plays an important role in order to prove the existence a regularity of u^*.

$$u_\lambda \text{ stable solution} \to \|u_\lambda\|_\infty \leq C.$$

The Problem:

Existence positive solutions of

$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = \lambda f(u)\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

where:

- $\lambda > 0$, f is *"strictly increasing"* and derivable in $[0, \infty)$ with $f(0) > 0$.
- g is a positive function such that $\limsup_{s\to\infty} g(s) < \infty$ and $e^{-G(s)} \in L^1(1, \infty)$, being $G(s) = \int_0^s g(t)dt$.

We are thinking in:

$$f(s) = e^s,\ (1+s)^p\,, \ldots \qquad g(s) = c,\ \frac{1}{s^\gamma}, \ldots \quad c > 0, \gamma \in [0, 1).$$

why $g(u)|\nabla u|^2$?

Consider the functional $J : W_0^{1,2}(\Omega) \to \mathbb{R}$

$$J(v) = \frac{1}{2}\int_\Omega a(x)|\nabla v|^2 - \int_\Omega f\, v,$$

$0 < \alpha \leq a(x) \leq \beta, f \in L^{\frac{2N}{N+2}}(\Omega)$. $\exists\, u \in W_0^{1,2}(\Omega)$ minimum. Moreover, u is a solution of the E-L equation

$$v \in W_0^{1,2}(\Omega) : \ -\mathrm{div}(a(x)\nabla v) = f(x).$$

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The E-L equation associated is

$$-\mathrm{div}(a(v)\nabla v) + \frac{1}{2}a'(v)|\nabla v|^2 = f.$$

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$$u \in W_0^{1,2}(\Omega): \; -\Delta u + g(u)|\nabla u|^2 = h(x,u)$$

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- Extensions to systems of elliptic partial differential equations, unbounded domains, parabolic case.
- In recent years the case g singular appears, whose simple model is $g(s) = \frac{1}{s^\gamma}$. (Arcoya, Boccardo, Carmona, Leonori, Martínez-Aparicio, Rossi,...).
- Several applications: *growth patterns in clusters and fronts of solidification, growth of tumors, flame propagation,...*(Kardar, Parisi, Zhang, Berestychi, Kamin, ...).

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$$\begin{cases} -\Delta u + g(u)|\nabla u|^2 = \lambda\, f(u)\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

where:

- $\lambda > 0$, f derivable in $[0, \infty)$ with $f(0) > 0$ and increasing conditon $f'(s) - g(s)f(s) > 0$.
- $\limsup_{s\to\infty} g(s) < \infty$, $e^{-G(s)} \in L^1(1,\infty)$, being $G(s) = \int_0^s g(t)dt$.

Some Definitions

We recall that a function $0 < u \in W_0^{1,2}(\Omega)$ is a (weak) solution if $g(u)|\nabla u|^2,\ f(u) \in L^1(\Omega)$ and it satisfies

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega g(u)\,|\nabla u|^2\,\phi = \int_\Omega \lambda\, f(u)\,\phi, \tag{3}$$

for all test function $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

- If $u \leq v$ for every another solution v, we say that u is minimal.
- If u belongs to $L^\infty(\Omega)$ we say that u is bounded (or regular).
- Let u be a solution, we say that u is stable if $f'(u) - g(u)f(u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega |\nabla \phi|^2 \geq \lambda \int_\Omega (f'(u) - g(u)f(u))\phi^2$$

holds for every $\phi \in \mathcal{C}_c^\infty(\Omega)$.

Theorem (Crandall-Rabinowitz version)

There exists $0 < \lambda^* < \infty$ *such that there is a bounded minimal solution* w_λ *for every* $\lambda < \lambda^*$ *and no solution for* $\lambda > \lambda^*$*. Moreover, the sequence* $\{w_\lambda\}$ *is increasing respect to* λ*.*

Theorem (Crandall-Rabinowitz version)

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Theorem (Existence and regularity extremal solution)

Set $h(s) := e^{-G(s)} f(s)$ and assume that $\lim_{s\to\infty} \frac{sh'(s)}{h(s)} \in (1, \infty]$. Then

- *$u^*(x) := \lim_{\lambda\to\lambda^*} w_\lambda(x)$ is solution.*
- *u^* is bounded whenever*

$$N < \frac{4 + 2(\mu + \alpha) + 4\sqrt{\mu + \alpha}}{1 + \alpha},$$

$$\text{being} \quad \alpha = \lim_{s\to\infty} \frac{g(s)h(s)}{h'(s)}, \quad \mu = \lim_{s\to\infty} \frac{(h'(s))^2}{h''(s)h(s)}.$$

Corollary (Exponential version)

Let $g(s) = c > 0$ and $f(s) = e^{(c+1)s}$. Then,

- *There exists $0 < \lambda^* < \infty$ such that there is a bounded minimal solution w_λ of*

$$\begin{cases} -\Delta u + c|\nabla u|^2 = \lambda\, e^{(c+1)u}\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

  for every $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$.

- *u^* is solution.*
- *u^* is bounded whenever*

$$N < \frac{4 + 2(1+c) + 4\sqrt{1+c}}{1+c},$$

  $(\alpha = c, \mu = 1)$.

Corollary (Mignot-Puel version)

Let $g(s) = c > 0$ *and* $f(s) = e^{cs}(1+s)^p$, $p > 1$. *Then,*

- *There exists* $0 < \lambda^* < \infty$ *such that there is a bounded minimal solution* w_λ *of*

$$\begin{cases} -\Delta u + c|\nabla u|^2 = \lambda\, e^{cu}(1+u)^p\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

  for every $\lambda < \lambda^*$ *and no solution for* $\lambda > \lambda^*$.
- u^* *is solution.*
- u^* *is bounded whenever*

$$N < 4 + 2\frac{p}{p-1} + 4\sqrt{\frac{p}{p-1}},$$

  $(\alpha = 0, \mu = \frac{p}{p-1})$.

Gelfand problem for 1-homogeneous p-laplacian

$$\begin{cases} -\Delta_p^N u = \lambda f(u)\,, & \text{in } \Omega, \\ u = 0\,, & \text{on } \partial\Omega, \end{cases}$$

f is a general continuous nonlinearity that verifies: $f(0) > 0$, increasing and $\frac{f(s)}{s} \geq k > 0$.

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $p \in [2, \infty]$ and the operator Δ_p^N is the called 1-homogeneous p-laplacian defined by

$$\Delta_p^N u := \frac{1}{p-1} |\nabla u|^{2-p} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = \alpha \Delta u + \beta \Delta_\infty u, \qquad (4)$$

being

$$\alpha = \frac{1}{p-1}, \qquad \beta = \frac{p-2}{p-1} \qquad \text{and} \qquad \Delta_\infty u = \frac{\nabla u}{|\nabla u|} \cdot \left(D^2 u \frac{\nabla u}{|\nabla u|}\right)$$

is the 1-homogeneous infinity laplacian.

Main difficulties

- Operator in no-divergence form $\to$ framework of viscosity solutions.

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- Operator in no-divergence form $\rightarrow$ framework of viscosity solutions.
- Singular at $|\nabla u| = 0$ $\rightarrow$ semicontinuous envelopes.
- Lack of uniform ellipticity $\rightarrow$ extensive literature can not be applied.
- Lack of variational structure $\rightarrow$ stable solutions make no sense.

Results so far...

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Theorem 1

There exists a positive extremal parameter $\lambda^* = \lambda^*(\Omega, N, p)$ such that:

- If $\lambda < \lambda^*$, problem admits a minimal positive solution w_λ.
- If $\lambda > \lambda^*$, problem has no positive solution.

Moreover, the branch of minimal solutions $\{w_\lambda\}$ is increasing with λ.
In the case of a ball, $\Omega = B_r$, the minimal solution is radial.

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Theorem 1

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Theorem 2

For every fixed $p \in [2, \infty]$, there exists an unbounded continua of solutions $\mathcal{C}$ that emanates from $\lambda = 0, u = 0$, with $\mathcal{C} \subset [0, \lambda^*] \times \mathcal{C}(\overline{\Omega})$, being λ^* the extremal parameter. Moreover, for every fixed small λ there exists a continua of solutions $\mathcal{D} \subset [2, \infty] \times \mathcal{C}(\overline{\Omega})$, with

$$\|u\|_\infty \leq C, \qquad \forall p \in [2, \infty].$$

Thank you for your attention !