Solutions to Gelfand problems

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1 Classical Gelfand Problem

\[ \begin{cases} 
-\Delta u = \lambda f(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega, 
\end{cases} \]

2 Elliptic eq. with natural growth in the quadratic gradient term

\[ \begin{cases} 
-\Delta u + g(u)|\nabla u|^2 = \lambda f(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega, 
\end{cases} \]


3 1-homogeneous p-laplacian

\[ \begin{cases} 
-\Delta^N_p u = \lambda f(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega, 
\end{cases} \]

Work in preparation with J. Carmona and J.D. Rossi.
Existence of positive solutions

\[
\begin{cases}
-\Delta u = \lambda e^u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
\end{cases}
\]

(1)

- Model for the thermal reaction process in a combustible.
Classical Gelfand Problem

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u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
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(1)

- Model for the thermal reaction process in a combustible.
- No solutions for \( \lambda \) large. Even in the weak sense.
Ω = B₁(0): Gidas-Ni-Nirenberg → solutions are radially, symmetric and satisfy

\[
\begin{align*}
-u'' - \frac{N-1}{r}u' &= \lambda e^u, \quad r \in [0, 1), \\
u'(0) &= u(1) = 0,
\end{align*}
\]  

(2)

where \( u(r) := u(|x|) \).

- \( 1 \leq N \leq 2 \): There exists \( \lambda^* > 0 \) such that (2) has exactly one solution for \( \lambda = \lambda^* \) and exactly two solutions for each \( \lambda \in (0, \lambda^*) \).

- \( 2 < N < 10 \): Eq. (2) has a continuum of solutions which oscillates around the line \( \lambda = 2(N - 2) \); with the amplitude of oscillations tending to zero, as \( u(0) = \|u\|_{\infty} \to \infty \).

- \( N \geq 10 \): Eq. (2) has a unique solution for each \( \lambda \in (0, 2(N - 2)) \) and no solutions for \( \lambda \geq 2(N - 2) \): Moreover, \( \|u\|_{\infty} \to \infty \) as \( \lambda \to 2(N - 2) \).
Classical Gelfand Problem

Classical Gelfand type problems

\[
\begin{aligned}
-\Delta u &= \lambda f(u), \quad \text{in } \Omega, \\
u &> 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]  

\( (G_\lambda) \)

\( \Omega \subset \mathbb{R}^N \) bounded, smooth, \( \lambda > 0, f(0) > 0 \), derivable, increasing, convex and superlinear \( \left( \lim_{s \to \infty} \frac{f(s)}{s} = \infty \right) \).

\( f(u) = e^u, (1 + u)^p, \frac{1}{(1-u)^k} \ldots \) with \( p > 1, k > 0 \).

Crandall-Rabinowitz (1973)

There exists a positive number \( \lambda^* \) called the extremal parameter such that

- If \( \lambda < \lambda^* \) the problem \((G_\lambda)\) admits a minimal bounded solution \( w_\lambda \).
- If \( \lambda > \lambda^* \) the problem \((G_\lambda)\) admits no solution.

Even more, the sequence \( \{w_\lambda\} \) is increasing.
Classical Gelfand type problems

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(G\(_\lambda\))

\(\Omega \subset \mathbb{R}^N\) bounded, smooth, \(\lambda > 0\), \(f(0) > 0\), derivable, increasing, convex and superlinear \(\left( \lim_{s \to \infty} \frac{f(s)}{s} = \infty \right)\).

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There exists a positive number \(\lambda^*\) called the \textit{extremal parameter} such that

- If \(\lambda < \lambda^*\) the problem \((G\_\lambda)\) admits a minimal bounded solution \(w\_\lambda\).
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Even more, the sequence \(\{w\_\lambda\}\) is increasing.
Classical Gelfand Problem

What happens when $\lambda = \lambda^*$?

Let $u^*(x) := \lim_{\lambda \to \lambda^*} w_\lambda(x)$
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Crandall-Rabinowitz (1975) $f(u) = (1 + u)^p$, $p > 1$

The *extremal solution* $u^*$ is bounded in dimensions

$$N < 4 + 2 \frac{p}{p - 1} + 4 \sqrt{\frac{p}{p - 1}}$$

for any domain $\Omega$. 

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Mignot-Puel (1980) $f(u) = e^u$

- The extremal solution $u^*$ is bounded in dimensions $N \leq 9$ for any domain $\Omega$.
- When $N \geq 10$, $u^*(x) = \log \frac{1}{|x|^2}$ is the singular extremal solution for $\Omega = B_1(0)$. 

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Brézis-Vázquez (1997)

What is the regularity of $u^*$ (depending on dimension N) for a general nonlinearities $f$?
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- $u^*$ is bounded for $N = 4$, S. Villegas (2013).
- Still unknown for $5 \leq N \leq 9$! There are some results....
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The stability condition, i.e., $u_\lambda$ satisfies

$$
\int_\Omega |\nabla \phi|^2 \geq \lambda \int_\Omega f'(u_\lambda) \phi^2, \quad \forall \phi \in C_0^\infty(\Omega),
$$

plays an important role in order to prove the existence a regularity of $u^*$.

$u_\lambda$ stable solution $\rightarrow \|u_\lambda\|_\infty \leq C$. 
The Problem:
Existence positive solutions of

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\begin{cases}
-\Delta u + g(u)|\nabla u|^2 = \lambda f(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
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\]

where:

- $\lambda > 0$, $f$ is "strictly increasing" and derivable in $[0, \infty)$ with $f(0) > 0$.
- $g$ is a positive function such that $\limsup_{s \to \infty} g(s) < \infty$ and $e^{-G(s)} \in L^1(1, \infty)$, being $G(s) = \int_0^s g(t)dt$.

We are thinking in:

\[
f(s) = e^s, (1 + s)^p, \ldots \quad g(s) = c, \frac{1}{s^\gamma}, \ldots \quad c > 0, \gamma \in [0, 1).
\]
why \( g(u)|\nabla u|^2 \) ?

Consider the functional \( J : W_0^{1,2}(\Omega) \to \mathbb{R} \)

\[
J(v) = \frac{1}{2} \int_\Omega a(x)|\nabla v|^2 - \int_\Omega f v,
\]

\( 0 < \alpha \leq a(x) \leq \beta, \ f \in L^{\frac{2N}{N+2}}(\Omega). \) \( \exists u \in W_0^{1,2}(\Omega) \) minimum. Moreover, \( u \) is a solution of the E-L equation

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v \in W_0^{1,2}(\Omega) : -\text{div}(a(x)\nabla v) = f(x).
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\]

The E-L equation associated is

\[-\text{div}(a(v)\nabla v) + \frac{1}{2} a'(v)|\nabla v|^2 = f.\]
Elliptic equation with natural growth in the quadratic gradient term

\[ u \in W_0^{1,2}(\Omega) : -\Delta u + g(u)|\nabla u|^2 = h(x, u) \]
Elliptic eq. with natural growth in the quadratic gradient term

\[ u \in W^{1,2}_0(\Omega) : -\Delta u + g(u)|\nabla u|^2 = h(x, u) \]

• Invariant to non-linear changes of variable \( v = F(u) \) \((F \in C^1)\).

• Non-existence result when the growth in \( \nabla u \) is faster than quadratic at infinity (Serrin'69).

• Existence for \( g \) continuous by Boccardo-Murat-Puel '82 and an extensive literature since then: Abdellaoui, Arcoya, Bensoussan, Dall'Aglio, Gallouët, Peral, Giachetti, Segura de León,...

• Extensions to systems of elliptic partial differential equations, unbounded domains, parabolic case.

• In recent years the case \( g \) singular appears, whose simple model is \( g(s) = \frac{1}{s^\gamma} \). (Arcoya, Boccardo, Carmona, Leonori, Martínez-Aparicio, Rossi,...).

• Several applications: growth patterns in clusters and fronts of solidification, growth of tumors, flame propagation,... (Kardar, Parisi, Zhang, Berestychi, Kamin, ...).
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where:
- \(\lambda > 0\), \(f\) derivable in \([0, \infty)\) with \(f(0) > 0\) and increasing condition \(f'(s) - g(s)f(s) > 0\).
- \(\lim \sup_{s \to \infty} g(s) < \infty\), \(e^{-G(s)} \in L^1(1, \infty)\), being \(G(s) = \int_0^s g(t)dt\).
Some Definitions

We recall that a function $0 < u \in W^{1,2}_0(\Omega)$ is a (weak) solution if $g(u)|\nabla u|^2$, $f(u) \in L^1(\Omega)$ and it satisfies

$$
\int_\Omega \nabla u \nabla \phi + \int_\Omega g(u) |\nabla u|^2 \phi = \int_\Omega \lambda f(u) \phi,
$$

for all test function $\phi \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$.

- If $u \leq v$ for every another solution $v$, we say that $u$ is minimal.
- If $u$ belongs to $L^\infty(\Omega)$ we say that $u$ is bounded (or regular).
- Let $u$ be a solution, we say that $u$ is stable if $f'(u) - g(u)f(u) \in L^1_{loc}(\Omega)$ and

$$
\int_\Omega |\nabla \phi|^2 \geq \lambda \int_\Omega (f'(u) - g(u)f(u))\phi^2
$$

holds for every $\phi \in C^\infty_c(\Omega)$.
Elliptic eq. with natural growth in the quadratic gradient term

Theorem (Crandall-Rabinowitz version)

There exists $0 < \lambda^* < \infty$ such that there is a bounded minimal solution $w_\lambda$ for every $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$. Moreover, the sequence $\{w_\lambda\}$ is increasing respect to $\lambda$. 

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Theorem (Crandall-Rabinowitz version)

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Theorem (Existence and regularity extremal solution)

Set $h(s) := e^{-G(s)} f(s)$ and assume that $\lim_{s \to \infty} \frac{sh'(s)}{h(s)} \in (1, \infty]$. Then

- $u^*(x) := \lim_{\lambda \to \lambda^*} w_\lambda(x)$ is solution.
- $u^*$ is bounded whenever

$$N < \frac{4 + 2(\mu + \alpha) + 4\sqrt{\mu + \alpha}}{1 + \alpha},$$

being $\alpha = \lim_{s \to \infty} \frac{g(s)h(s)}{h'(s)}$, $\mu = \lim_{s \to \infty} \frac{(h'(s))^2}{h''(s)h(s)}$. 
Corollary (Exponential version)

Let $g(s) = c > 0$ and $f(s) = e^{(c+1)s}$. Then,

- There exists $0 < \lambda^* < \infty$ such that there is a bounded minimal solution $w_\lambda$ of

\[
\begin{cases}
-\Delta u + c|\nabla u|^2 = \lambda e^{(c+1)u}, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

for every $\lambda < \lambda^*$ and no solution for $\lambda > \lambda^*$.

- $u^*$ is solution.

- $u^*$ is bounded whenever

\[
N < \frac{4 + 2(1 + c) + 4\sqrt{1 + c}}{1 + c},
\]

$(\alpha = c, \mu = 1)$. 

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Solutions to Gelfand problems
Corollary (Mignot-Puel version)

Let \( g(s) = c > 0 \) and \( f(s) = e^{cs}(1 + s)^p, \ p > 1 \). Then,

- There exists \( 0 < \lambda^* < \infty \) such that there is a bounded minimal solution \( w_\lambda \) of

\[
\begin{align*}
-\Delta u + c|\nabla u|^2 &= \lambda e^{cu}(1 + u)^p, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

for every \( \lambda < \lambda^* \) and no solution for \( \lambda > \lambda^* \).

- \( u^* \) is solution.

- \( u^* \) is bounded whenever

\[
N < 4 + 2 \frac{p}{p - 1} + 4 \sqrt{\frac{p}{p - 1}},
\]

\((\alpha = 0, \mu = \frac{p}{p-1})\).
1-homogeneous p-laplacian

Gelfand problem for 1-homogeneous p-laplacian

\[
\begin{aligned}
-\Delta^N_p u &= \lambda f(u), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

\(f\) is a general continuous nonlinearity that verifies: \(f(0) > 0\), increasing and \(\frac{f(s)}{s} \geq k > 0\).

where \(\Omega \subset \mathbb{R}^N\) is a regular bounded domain, \(p \in [2, \infty]\) and the operator \(\Delta^N_p\) is the called 1-homogeneous p-laplacian defined by

\[
\Delta^N_p u := \frac{1}{p-1} |\nabla u|^{2-p} \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = \alpha \Delta u + \beta \Delta_{\infty} u,
\]

being

\[
\alpha = \frac{1}{p-1}, \quad \beta = \frac{p-2}{p-1}
\]

and

\[
\Delta_{\infty} u = \frac{\nabla u}{|\nabla u|} \cdot \left( D^2 u \frac{\nabla u}{|\nabla u|} \right)
\]

is the 1-homogeneous infinity laplacian.

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Main difficulties

- Operator in no-divergence form $\rightarrow$ framework of viscosity solutions.
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- Singular at $|\nabla u| = 0$ $\rightarrow$ semicontinuous envelopes.
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- Singular at $|\nabla u| = 0$ $\rightarrow$ semicontinuous envelopes.
- Lack of uniform ellipticity $\rightarrow$ extensive literature cannot be applied.
- Lack of variational structure $\rightarrow$ stable solutions make no sense.
1-homogeneous p-laplacian

Results so far...

Theorem 1
There exists a positive extremal parameter $\lambda^*$ such that:
- If $\lambda < \lambda^*$, problem admits a minimal positive solution $w_\lambda$.
- If $\lambda > \lambda^*$, problem has no positive solution.

Moreover, the branch of minimal solutions $\{w_\lambda\}$ is increasing with $\lambda$.
In the case of a ball, $\Omega = B_r$, the minimal solution is radial.

Theorem 2
For every fixed $p \in [2, \infty]$, there exists an unbounded continua of solutions $C$ that emanates from $\lambda = 0, u = 0$, with $C \subset [0, \lambda^*] \times C(\Omega)$, being $\lambda^*$ the extremal parameter. Moreover, for every fixed small $\lambda$ there exists a continua of solutions $D \subset [2, \infty] \times C(\Omega)$, with $\|u\|_\infty \leq C, \forall p \in [2, \infty]$. 
1-homogeneous p-laplacian

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Theorem 1

There exists a positive extremal parameter $\lambda^* = \lambda^*(\Omega, N, p)$ such that:

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$$\|u\|_\infty \leq C, \quad \forall p \in [2, \infty].$$
Thank you for your attention!