Aspects of the Unitarity Approach for Multi-Loop Amplitudes in QCD

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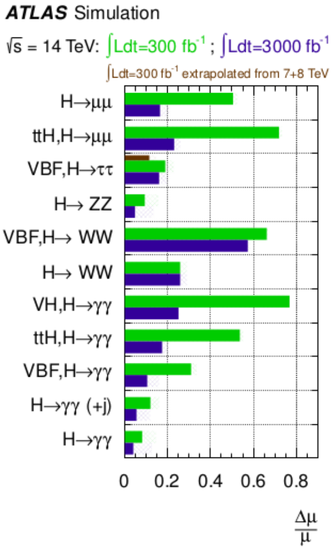
Based on work with F. Febres-Cordero, H. Ita, M. Jaquier and M. Zeng.

LHC Era Phenomenology

- High-lumunisoty run of LHC will substantially improve experimental precision.
- $\Rightarrow$ Need for NNLO predictions for many processes.



Progress in NNLO Phenomenology

- Di-photon: [Catani et al 11, Campbell et al 16]
- Dijet: [Currie et al 14]
- W+J: [Boughezal et al 15]
- Z+J: [Gehrmann-De Ridder et al 15, Boughezal et al 15]
- H+J: [Chen et al 16, Caola et al 15, Boughezal et al 15]
- tt: [Czakon et al 16]
- WW: [Gehrmann et al 14, Caola et al 15]
- ZZ: [Cascioli et al 14, Grazzini et al 15, Melnikov et al 16]
- ZH: [Ferrera et al 14, Campbell et al 16]
- Zγ, Wγ: [Grazzini et al 14]
- HH: [de Florian et al 16]

Hopes for Beyond 2 $\rightarrow$ 2?

- 5/6-point QCD amplitudes: [Badger et al 15, Gehrmann et al 15, Dunbar et al 16, Badger et al 16]

A Bottleneck: Two Loop Amplitude Calculations

Feynman diagrams

Tensor reduction
[Tarasov 96; Anastasiou, Glover, Oleari 99]
IBPs
[Tkachov, Chetyrkin 81]
↓

Sum of master integrals

$$\mathcal{A} = \sum_i c_i I_i^{\text{master}}$$

Differential equations
[Gehrmann, Remiddi 01]
(see Wever talk)
↓

Integrated form

- Standard procedure.
- Process specific.
- Can we repeat the leap made at NLO?

Numerical Unitarity for Automatic Amplitudes

AIM: Write amplitude ($\mathcal{A}$) as a sum of master integrals.

$$\mathcal{A} = \int A = \int \sum_i \frac{N_i}{\rho^1 \ldots \rho^{n_i}} = \sum_i c_i \int \frac{t_i^{\mathrm{master}}}{\rho^1 \ldots \rho^{n_i}}$$

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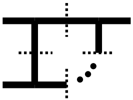
General algorithm:



$$\overset{\text{Unitarity}}{\longleftrightarrow} \quad \underset{\{\rho^1,\cdots,\rho^{n_i}\}=0}{\text{Residue}}(A) \quad \overset{\text{Subtraction}}{\longleftrightarrow} \quad N_i = \sum_i c_i t_i$$

[Bern, Dixon, Kosower]

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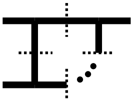

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2-loop complications:

- IBPs - how to find basis $\{t_i^{\text{master}}, t_i^{\text{surface}}\}$? [Ita 15]
- Double propagators - residue $\leftrightarrow$ product of trees?

A Helpful Coordinate System for IBPs

Parameterize in inverse propagators (ρ), transverse variables (α).

$$\ell^\mu = \sum_{i=1}^{D_p} r_i(\rho) v_i^\mu + \sum_{a=1}^{D_t} \alpha_a n_a^\mu, \qquad r_i = (\ell \cdot p_i).$$

[van Neerven, Vermaseren 84; Ellis, Giele, Kunst 07]

- v_i dual directions to p_i, n_i mutually orthonormal directions.
- On-shell conditions manifest ($\rho^i \to 0$).
- Extra degree of freedom controlled by constraint:

$$c(\rho, \alpha) = -m_0^2 - \rho^0 + \left(\sum_{i=1}^{D_p} r_i v_i^\mu \right)^2 + \sum_{a=1}^{D_t} (\alpha^a)^2 = 0.$$

IBPs in Adapted Coordinates

- Objects of study are IBP vectors $\bar{u}^\mu$: [Gluza, Kadja, Kosower 11]

$$\int d^d\ell \left\{\partial_\mu \left(\frac{\bar{u}^\mu}{\rho^1 \cdots \rho^n}\right)\right\} = \int [d\rho^i][d\alpha^a] \left\{\partial_i \left(\frac{\bar{u}^i}{\rho^1 \cdots \rho^n}\right) + \frac{\partial_a \bar{u}^a}{\rho^1 \cdots \rho^n}\right\}$$

- Impose $\bar{u}^\mu \partial_\mu c = 0$ to remain in physical phase space. [Ita 15]

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$$(\bar{u}^i \partial_i)\rho^k = \bar{u}_k = f^k \rho^k,$$ [Larsen, Zhang 15; Ita 15]

$$\Rightarrow \partial_\mu \left(\frac{\bar{u}^\mu}{\rho^1 \cdots \rho^n} \right) = \frac{\partial_i \bar{u}^i}{\rho^1 \cdots \rho^n} + \frac{\partial_a \bar{u}^a}{\rho^1 \cdots \rho^n} - \sum_j \frac{f^j}{\rho^1 \cdots \rho^n}.$$

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- Cleverly chosen $\bar{u}^\mu \Rightarrow$ vanishing numerator combinations.

1-loop IBP Generating Vectors

[Ita 15]

- Identify generating vectors u^μ such that $\bar{u}^\mu = t(\rho, \alpha) u^\mu$.
- Useful categorizations of $u^\mu \partial_\mu$:

Horizontal ($\rho = 0$)	$u_{[ab]} = \alpha^a \partial_b - \alpha^b \partial_a$	Build surface terms
Vertical ($\alpha_i = 0$)	$\sum_i f^i \rho^i \partial_i$	Link topologies
Scaling	$\sum_i f^i \rho^i \partial_i + \sum_a g^a \alpha^a \partial_a$	Singular transverse space

Surface terms easily enumerated:

- Write all independent $\bar{u}^\mu = t(\alpha) u^\mu$ satisfying power counting.
- Insert vector into IBP equation.
- Simply gives standard 1-loop decomposition.

Horizontal Two-loop IBP Generating Vectors

[Ita 15]

- Build two-loop coordinates from multiple one-loop systems.
- 2-loop adapted coordinates must satisfy $\{c, \tilde{c}, \hat{c}\} = 0$.
- Generating vectors satisfy $u^\mu \partial_\mu \{c, \tilde{c}, \hat{c}\} = 0$.

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2-Loop Horizontal Generators $(u^\mu \partial_\mu)$:

Axis rotations: $u_{[abc]} = (\partial_{[a|} \hat{c}) u_{|bc]}$

Diagonal rotations: $u^{\text{diag}}_{[ab]} = u_{[ab]} + \tilde{u}_{[ab]}$

Crossed rotations: $u_{[ab][cd]} = \left(\tilde{u}_{[cd]} \hat{c}\right) u_{[ab]} - \left(u_{[ab]} \hat{c}\right) \tilde{u}_{[cd]}$

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Can find master/surface basis: $\{t_i\} \to \{\{t_i^{\mathrm{master}}\}, \{t_i^{\mathrm{surface}}\}\}$.

For a related approach see [Mastrolia et al 16]

Ready to Construct an Amplitude?

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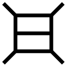
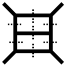
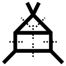
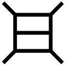
$N\Big($  $\Big) =$  $= \sum_i c_i t_i^{db,\mathrm{master}} + \sum_j c_j t_j^{db,\mathrm{surface}}$

$N\Big($  $\Big) =$  $- N\Big($  $\Big) = \sum_i d_i t_i^{tb,\mathrm{master}} + \sum_j d_j t_j^{tb,\mathrm{surface}}$

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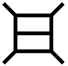
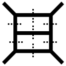
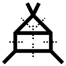
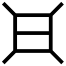
$N\Big($  $\Big) =$  $= \sum_i c_i t_i^{db,\text{master}} + \sum_j c_j t_j^{db,\text{surface}}$

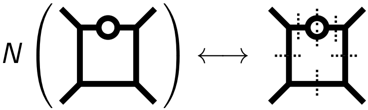
$N\Big($  $\Big) =$  $- N\Big($  $\Big) = \sum_i d_i t_i^{tb,\text{master}} + \sum_j d_j t_j^{tb,\text{surface}}$

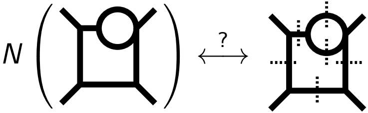
However: Procedure not sufficient given double propagators.

Double Propagators ⇒ Undefined Cuts

[Page et al, WIP]

Double propagator breaks standard relation between cut and N:





Ill-defined.

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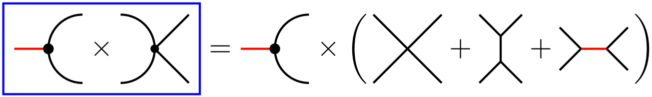
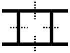
One tree amplitude contains the on-shell propagator:

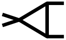


Must obtain this numerator some other way.

Numerators from Lower Cuts - “Cutting less”

Standard approach

$N\Big($  $\Big) =$  $= \sum_i c_i^b t_i^b$

$N\Big($  $\Big) =$  $- \sum_i c_i^b \frac{t_i^b}{\rho} = \sum_i c_i^t t_i^t$

1. Calculate c_i^b on cut.
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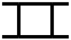
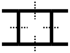
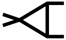
$N\Big($  $\Big) =$  $= \sum_i c_i^b t_i^b$

$N\Big($  $\Big) =$  $- \sum_i c_i^b \frac{t_i^b}{\rho} = \sum_i c_i^t t_i^t$

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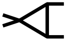
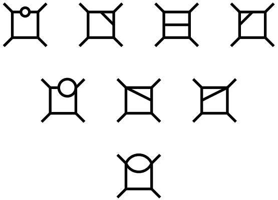
vs "Cut less"

 $= N\Big($  $\Big) + \frac{1}{\rho} N\Big($  $\Big)$

$$= \sum_i c_i^t t_i^t + \sum_i c_i^b \frac{t_i^b}{\rho}$$

1. Calculate together, avoiding higher cut.

“Cutting less” for Troublesome Topologies



Subtraction hierarchy (1).

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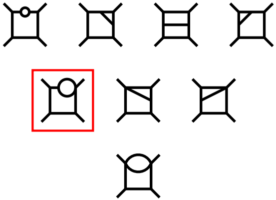

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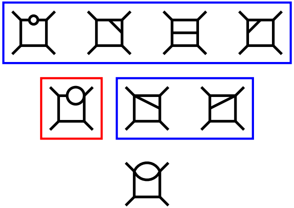

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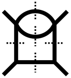
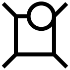
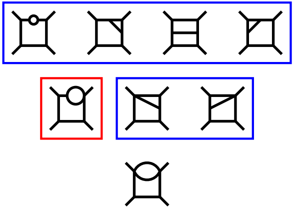
 $- \sum_{\text{parents}} \frac{N_i}{\rho^i} - \sum_{\text{gparents}} \frac{N_{ij}}{\rho^i \rho^j}$

$= N\Big($  $\Big) + \frac{1}{\rho} N \Big($  $\Big)$

$$= \sum_i c_i t_i^a + \sum_i d_i \frac{1}{\rho} t_i^{\text{troublesome}}.$$



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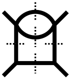
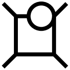
 $- \sum_{\text{parents}} \frac{N_i}{\rho^i} - \sum_{\text{gparents}} \frac{N_{ij}}{\rho^i \rho^j}$

$= N\Big($  $\Big) + \frac{1}{\rho} N\Big($  $\Big)$

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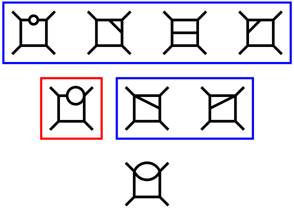

Subtraction hierarchy (1).

$N\,($  $)$ by cutting less: status

- Achieved tensor basis fit of hierarchy (1) including $N\,($  $)$.
- WIP: fit onto master/surface basis.

Conclusions

- Numerical unitarity is a valuable tool for phenomenologically relevant calculations.
- At two loop new technical difficulties arise:
  - Master/surface numerator basis - handled by classifying IBPs.
  - Double propagators - handled by cutting less.
- We present solutions to each of these problems and apply them in particular to double propagator child topologies.