Branching Brownian particles with spatial selection and the KPP equation

Julián Martínez
joint work with Pablo Groisman y Matthieu Jonckheere.

Fisher - Kolmogorov, Petrovskii, Piskunov equation

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Infinitely many travelling waves

$$u(x,t) = w(x-ct), \qquad \text{with } c = c(f).$$

But only one has physical meaning, $c = \sqrt{2}$.

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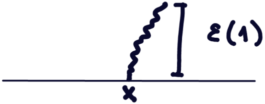

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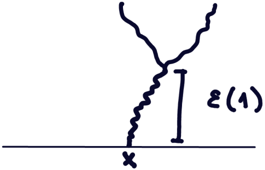

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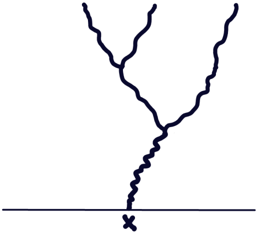

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- Brunet, Derrida; 1997,2001,...:
  Efect of microscopic noise in front propagation,
  mainly numerical and heuristic arguments.

$$v_N - v_{min} \simeq \frac{-K}{\log^2 N}$$

Our model

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We are interested in $$\mathbf{F_N(x,t)} = \frac{\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{1}_{\xi_{\mathbf{t}}^{\mathbf{i}} \leq \mathbf{x}}}{\mathbf{N}}.$$

LGN: $(\xi^i)_{i\geq 1}, \quad \xi^i \sim X, i.i.d \quad \Longrightarrow \qquad F_N(x) \xrightarrow[N\to\infty]{P} P(X \leq x) \qquad \forall x.$

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Hydrodynamic limit

If $\xi_0^N \sim \rho_0^N$ such that $F_N(x,0) \xrightarrow[N\to\infty]{P} u_0(x) \quad \forall x,$ then

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$$\begin{aligned}\partial_t u_N &= \overbrace{\partial_{xx} u_N}^{Difussion} - \overbrace{\tfrac{N}{N-1} E[F_N(1-F_N)]}^{jumps}\\ &= \partial_{xx} u_N - u_N(1-u_N) + \tfrac{N}{N-1} V(F_N(x,t)) + \tfrac{1}{N-1}(u_N(1-u_N)).\end{aligned}$$

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Variance / Propagation of chaos

$$F_N(x,t)^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} 1_{\xi_t^i \leq x} 1_{\xi_t^j \leq x}$$

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Lemma

$$\sup_{\bar{\xi}_0^N} \frac{1}{N^2}\left|\sum_{i=1}^{N}\sum_{j} P(\xi_t^i \le x, \xi_t^j \le x) - P(\xi_t^i \le x)P(\xi_t^j \le x)\right| \le \frac{2e^t}{N}.$$

Velocity - Upper bound

Idea: Embed our process in N independent BBM

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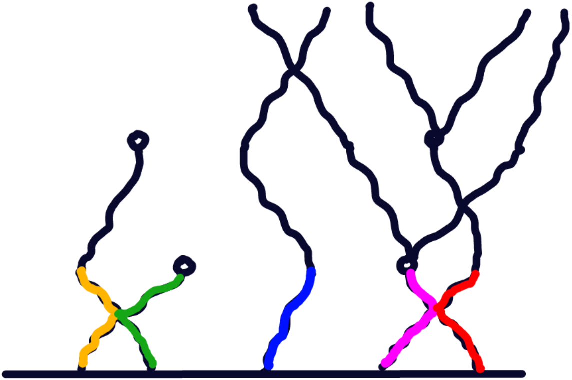

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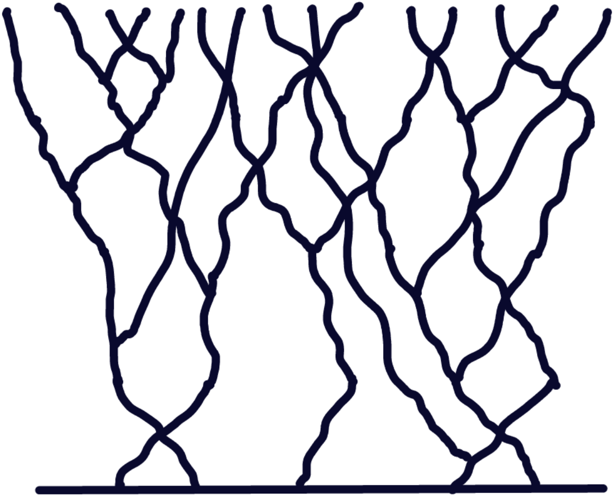

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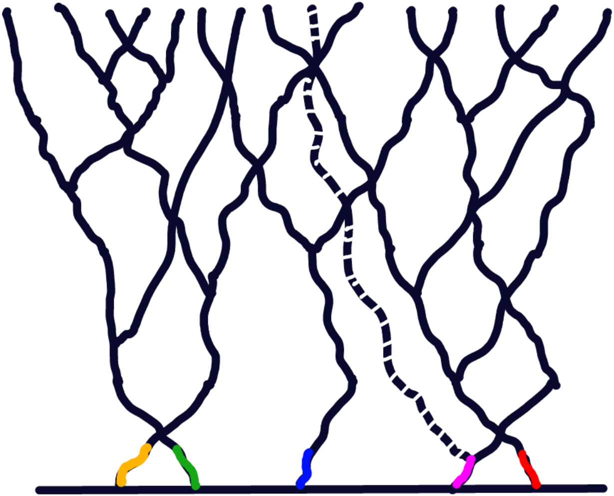

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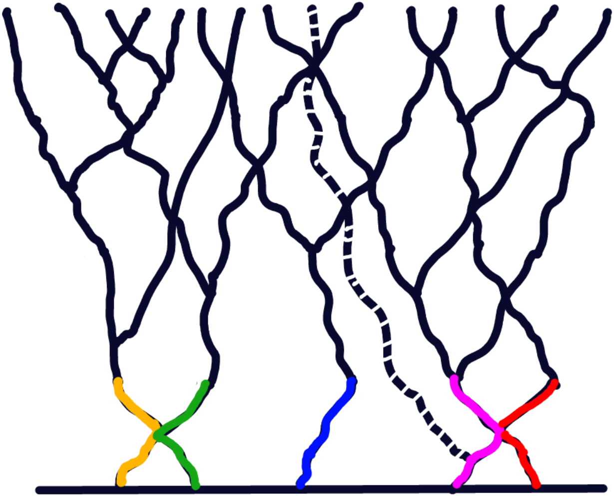

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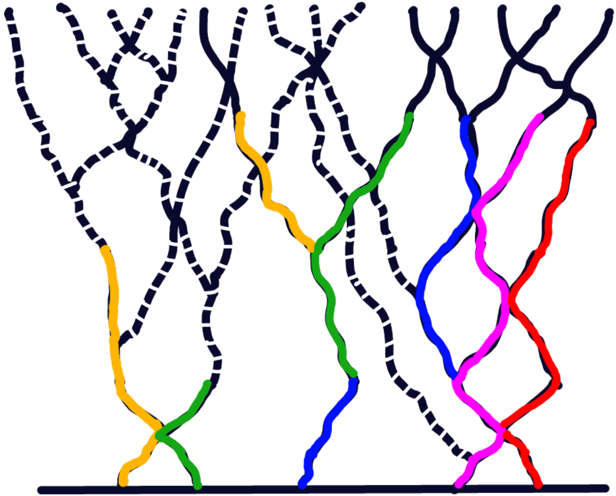

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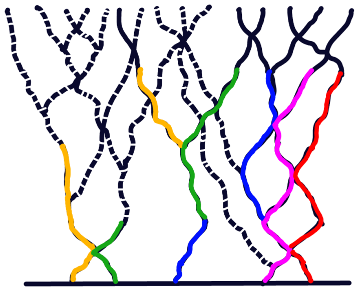

Maximum of the BBM

Let $M_t := \max\{X_t^1, \dots, X_t^{\mathcal{N}_t}\}$. Then

$$\lim_{t\to\infty} \frac{M_t}{t} = \sqrt{2}, \qquad c.s.$$

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$$\underset{(2)}{\Rightarrow} \qquad c_2 = v_N$$

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$$\lim_{t\to\infty}\lim_{b\to\infty}\int_{m(t)-b}^{m(t)+b} u^0(x,t)(1-u^0(x,t))dx = \sqrt{2}$$

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Thanks for your attention!