Dispersion property for Schrödinger equations

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Outline

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Linear Schrödinger equation

$$\begin{cases} iu_t(t,x) + u_{xx}(t,x) = 0, \quad t \in \mathbb{R}, x \in \mathbb{R} \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}. \end{cases}$$

1. Qualitative properties of the solutions
2. Decay of the solutions
3. Conservation of some quantities
4. Numerical approximations
5. The same equation on trees, graphs
6. Discrete models

Fourier Transform

Basic properties

1. $$\hat{u}(\xi) = \int_{\mathbb{R}} e^{-2i\pi x\xi} u(x)dx$$

2. $$u(x) = \int_{\mathbb{R}} e^{2i\pi x\xi} \hat{u}(\xi)d\xi$$

3. $$\int_{\mathbb{R}} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}} |u(x)|^2 dx$$

4. $$|\hat{u}(\xi)| \leq \int_{\mathbb{R}} |u(x)|dx$$

Using Fourier transform we get

$$\hat{u}(t,\xi) = e^{-it(2\pi\xi)^2}\hat{\varphi}(\xi)$$

and

$$u(t,x) = (K_t * \varphi)(x),$$

where

$$K_t(x) = \frac{e^{\frac{i|x|^2}{4t}}}{(4i\pi t)^{1/2}}$$

Two important properties

Conservation of the $L^2(\mathbb{R})$-norm

$$\int_{\mathbb{R}} |u(t,x)|^2 dx = \int_{\mathbb{R}} |\hat{u}(t,\xi)|^2 dx = \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 d\xi = \int_{\mathbb{R}} |\varphi(\xi)|^2 dx$$

or

$$\frac{d}{dt}\int_{\mathbb{R}} |u(t,x)|^2 dx = 2\Re \int_{\mathbb{R}} u_t(t,x)\overline{u(t,x)}dx = 0$$

Dispersive property

$$|u(t,x)| \leq \int_{\mathbb{R}} |K_t(x-y)||\varphi(y)|dy \lesssim \frac{1}{t^{1/2}} \int_{\mathbb{R}} |\varphi(y)|dy.$$

Nonlinear problems

Nonlinear problems are solved by using fixed point arguments on the variation of constants formulation of the PDE:

$$u_t(t) = Au(t) + f(u(t)),\ t > 0,\ u(0) = u_0.$$

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(u(s))ds.$$

Assuming $f : H \to H$ is locally Lipschitz, allows proving local existence and uniqueness in

$$u \in C([0, T]; H)$$

Ex: $H = H^1(\mathbb{R})$, $f(u) = |u|^2 u$

But, often in applications, $f : H \to H$ is not locally Lipshitz.

For instance $H = L^2(\mathbb{R})$ and $f(u) = |u|^p u$, with $p > 0$.

Then, one needs to discover other properties of the underlying linear equation (smoothing, dispersion): If $e^{At}\varphi \in X$, then look for solutions of the nonlinear problem in

$$C([0,T];H) \cap X.$$

One then needs to investigate whether

$$u \to e^{At}u_0 + \int_0^t e^{A(t-s)} f(u(s))ds$$

is a contraction in $C([0,T];H) \cap X$.

Typically in applications $X = L^q(0,T;L^r(\mathbb{R}))$. This allows enlarging the class of solvable nonlinear PDE in a significant way.

Linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = 0,\ x \in \mathbb{R},\ t \neq 0, \\ u(0,x) = \varphi(x),\ x \in \mathbb{R}, \end{cases}$$

Conservation of the L^2-norm

$$\|e^{it\Delta}\varphi\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$$

Dispersive estimate

$$\|e^{it\Delta}\varphi\|_{L^\infty(\mathbb{R})} = \|K_t * \varphi\|_{L^\infty(\mathbb{R})} \leq \frac{1}{(4\pi|t|)^{1/2}}\|\varphi\|_{L^1(\mathbb{R})}$$

Interpolation

$$\|e^{it\Delta}\varphi\|_{L^{p'}(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{p'})}\|\varphi\|_{L^p(\mathbb{R})},\ p \in [1,2]$$

Space time estimates

The admissible pairs

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}$$

Strichartz estimates for admissible pairs (q, r)

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}))} \leq C(q, r)\|\varphi\|_{L^2(\mathbb{R})}$$

Local Smoothing effect

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} ||\partial_x|^{1/2}(e^{it\Delta}\varphi)|^2 dt \leq C\|\varphi\|^2_{L^2(\mathbb{R})}$$

Nonlinear Schrödinger Equation

$$\begin{cases} iu_t + \Delta u = |u|^p u, \; x \in \mathbb{R}, \; t \neq 0 \\ u(0,x) = \varphi(x), \; x \in \mathbb{R} \end{cases}$$

For initial data in $L^2(\mathbb{R})$, Tsutsumi '87 proved the global existence and uniqueness for $p < 4$

$$u \in C(\mathbb{R}, \, L^2(\mathbb{R})) \cap L^q_{loc}(\mathbb{R}, \, L^r(\mathbb{R}))$$

Proof : Banach's fix point argument in balls of

$$C([0,T], \, L^2(\mathbb{R})) \cap L^q([0,T], \, L^r(\mathbb{R}))$$

A first numerical scheme for NSE

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = |u^h|^2 u^h, \quad t \neq 0, \\ \\ u^h(0) = \varphi^h \end{cases}$$

$$(\Delta_h u)_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

Questions

- Does u^h converge to the solution of NSE?
- Is u^h uniformly bounded in $L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$?
- Local Smoothing ?

A conservative scheme for LSE

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = 0, \quad t > 0, \\ u^h(0) = \varphi^h. \end{cases}$$

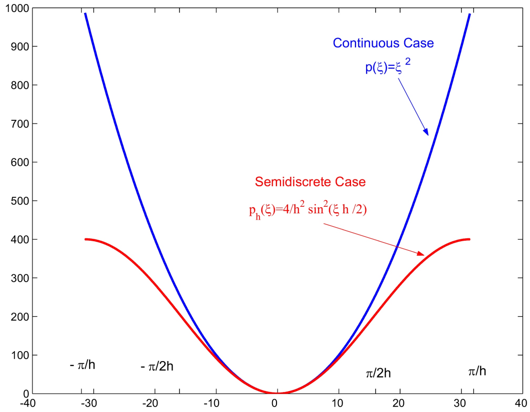
In the Fourier space the solution $\widehat{u}^h$ can be written as

$$\widehat{u}^h(t,\xi) = e^{-itp_h(\xi)}\widehat{\varphi}^h(\xi),\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right],$$

where

$$p_h(\xi) = \frac{4}{h^2}\sin^2\left(\frac{\xi h}{2}\right).$$

The two symbols in dimension one



- Lack of uniform $l^1 \to l^\infty$: $\xi = \pm\pi/2h$
- Lack of uniform local smoothing effect: $\xi = \pm\pi/h$

Lemma

(Van der Corput) Suppose ψ is real-valued and smooth in (a,b), and that $|\psi^{(k)}(x)| \geq 1$ for all $x \in (a,b)$. Then

$$\left| \int_a^b e^{i\lambda\psi(x)} dx \right| \leq c_k \lambda^{-1/k}$$

In dimension one:

$$\frac{\|u^h(t)\|_{l^\infty(h\mathbb{Z})}}{\|u^h(0)\|_{l^1(h\mathbb{Z})}} \lesssim \frac{1}{t^{1/2}} + \frac{1}{(th)^{1/3}}.$$

Various remedies have been proposed L.I and E. Zuazua (2003-2010)

- Filtering the high frequencies, Artificial numerical viscosity, Two-grid methods
- Error estimates for rough initial data
- Wave packet analysis, Wigner measure approach by A. Marica and E. Zuazua
- KdV by Corentin Audiard
- Discrete NLS with long-range lattice interactions by G. Staffilani
- Frequency saturation in NSE by Remi Carles,
- etc...

Outline

Discrete Schrödinger equations

We consider

$$\begin{cases} iu_t + \Delta_d u = 0, \quad j \in \mathbb{Z}, t \neq 0, \\ u(0) = \varphi. \end{cases} \tag{1}$$

where

$$(\Delta_d u)_j = u_{j+1} - 2u_j + u_{j-1}$$

Theorem (Stefanov 2005, LI & Zuazua 2005)

For any $\varphi \in l^1(\mathbb{Z})$ the following holds

$$\|u(t)\|_{l^\infty(\mathbb{Z})} \leq \langle t \rangle^{-1/3} \|\varphi\|_{l^1(\mathbb{Z})} \tag{2}$$

where $\langle t \rangle = t + 1$.

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where $\langle t \rangle = t + 1$.

A simple proof

$$u(t,j)=(K_t*\varphi)(j)=\sum_{k\in\mathbb{Z}}K_t(j-k)\varphi(k),$$

where

$$K_t(j)=\int_{-\pi}^{\pi}e^{-4it\sin^2\frac{\xi}{2}}e^{ij\xi}d\xi.$$

It remains to prove that

$$|K_t(j)|\leq t^{-1/3}.$$

Apply Van der Corput and the fact that $\psi=4\sin^2\frac{\xi}{2}+ij\xi/4t$ satisfies

$$|\psi''|+|\psi'''|\geq C>0.$$

DLSE with Dirichlet boundary condition

We consider the following equation

$$\begin{cases} iu_t(t,j) + (\Delta_d u)(t,j) = 0, & j \geq 1, \\ u(t,0) = 0, & \\ u(0,j) = \varphi(j), & j \geq 1. \end{cases} \tag{3}$$

In the matrix formulation we have $iU_t + AU = 0$ where

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}$$

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Theorem

For any $\varphi \in l^2(\mathbb{Z}^+)$ *there exists a unique solution* $u \in C([0,\infty), l^2(\mathbb{Z}^+))$ *of problem* (3) *given by the following formula*

$$u(t,j) = \sum_{k\geq 1} (K_t(j-k) - K_t(j+k))\varphi(k), \quad j \geq 1.$$

Moreover

$$\|u(t)\|_{l^\infty(\mathbb{Z}^+)} \leq \langle t\rangle^{-1/3} \|\varphi\|_{l^1(\mathbb{Z}^+)}.$$

Proof: Use odd extension of the function u to reduce the DLSE on the whole Z.

$$\tilde{u}(t,x) = -u(t,-x), x < 0$$

satisfies

$$i\tilde{u}_t(t,j) + \Delta_d \tilde{u}(t,j) = 0, j \in \mathbb{Z}$$

DLSE with Neumann boundary conditions

We consider the system

$$\begin{cases} iu_t(j) + (\Delta_d u)(j) = 0 & j \geq 1, \\ u(t,0) = u(t,1), & t > 0, \\ u(0,j) = \varphi(j), & j \geq 1. \end{cases} \tag{4}$$

In the matrix formulation we have $iU_t + AU = 0$ where

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}$$

Theorem

For any $\varphi \in l^2(\mathbb{Z}^+)$ *there exists a unique solution* $u \in C([0,\infty), l^2(\mathbb{Z}^+))$ *of problem* (4) *given by the following formula*

$$u(t, j+1) = \sum_{k \geq 1} (K_t(k-j-1) + K_t(k+j))\varphi(k).$$

Moreover

$$\|u(t)\|_{l^\infty(\mathbb{Z}^+)} \leq \langle t \rangle^{-1/3} \|\varphi\|_{l^1(\mathbb{Z}^+)}.$$

Proof: Use the even extension of u:

$$\tilde{u}(t,x) = u(t,-x), x < 0$$

Coupled DLSE

The equation we analyze is the following

$$\begin{cases} iu_t(j) + (\Delta_d u)(j) = 0 & j \leq -1, \\ iv_t(j) + (\Delta_d v)(j) = 0 & j \geq 1, \\ u(t,0) = v(t,0), & t > 0, \\ u(t,-1) - u(t,0) = v(t,0) - v(t,1), & t > 0 \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases} \tag{5}$$

Theorem

For any $\varphi \in l^2(\mathbb{Z}^*)$ *there exist a unique solution* $(u,v) \in C([0,\infty, l^2(\mathbb{Z}^*))$ *of equation* (5) *which satisfies the dispersive estimate*

$$\|(u,v)(t)\|_{l^\infty(\mathbb{Z}^*)} \leq c(t+1)^{-1/3} \|\varphi\|_{l^1(\mathbb{Z}^*)}. \tag{6}$$

A simple proof

Define

$$S(j) = \frac{v(j) + u(-j)}{2}, j \geq 0, \;\; D(j) = \frac{v(j) - u(-j)}{2}, j \geq 0.$$

Observe that

$$(u, v) = ((S - D)(-\cdot), S + D)$$

Key point: D and S satisfy two DLSE on the half line with Dirichlet, respectively Neumann, boundary condition:

$$\begin{cases} iD_t(j) + (\Delta_d D)(j) = 0 & j \geq 1, \\ D(t,0) = 0, & \\ D(0,j) = \frac{\varphi(j) - \varphi(-j)}{2}, & j \geq 1 \end{cases} \tag{7}$$

and

$$\begin{cases} iS_t(j) + (\Delta_d S)(j) = 0 & j \geq 1, \\ S(t,0) = S(t,1), & t > 0, \\ S(0,j) = \frac{\varphi(j) + \varphi(-j)}{2}, & j \geq 1. \end{cases} \tag{8}$$

Matrix formulation

Set $U=(u,v)^T$ where $u=(u(j))_{j\leq -1}$ and $v=(v(j))_{j\geq 1}$. It turns out that U solves the following system

$$\begin{cases} iU_t + AU = 0, \quad t>0, \\ U(0)=\varphi, \end{cases} \tag{9}$$

where the operator A is given by

$$A=\begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

Open Problem

How we can obtain the $l^1 - l^\infty$ property directly from the properties of the operator A?

Remarks: A is not a diagonal operator, so we cannot use the Fourier analysis to obtain a symbol for A and to use oscillatory integrals

A can be decomposed as $A = \Delta_d + B$ where

$$\Delta_d = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

The solution of (9) is given by $U(t) = e^{it(\Delta_d + B)}$. How we can use the dispersive properties of $e^{it\Delta_d}$ and some properties of B in order to prove the $l^1 - l^\infty$ estimate for U?

DLSE with "non-constant coefficients"

The model (D. Stan, L.I., JFAA 2011)

$$\begin{cases} iu_t(j) + b_1^{-2}(\Delta_d u)(j) = 0 & j \leq -1, \\ iv_t(j) + b_2^{-2}(\Delta_d v)(j) = 0 & j \geq 1, \\ u(t,0) = v(t,0), & t > 0, \\ b_1^{-2}(u(t,-1) - u(t,0)) = b_2^{-2}(v(t,0) - v(t,1)), & t > 0 \\ u(0,j) = \varphi(j), & j \leq -1, \\ v(0,j) = \varphi(j), & j \geq 1. \end{cases}$$

Question: $\|(u,v)(t)\|_\infty \leq (1+|t|)^{-1/3}\|\varphi\|_{l^1(\mathbb{Z}^*)}$

Matrix formulation

$U = (u(j))_{j\neq 0}$ satisfies $iU_t + AU = 0$ where A is given by

$$\begin{pmatrix} \dots & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1^{-2} & -2b_1^{-2} & b_1^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_1^{-2} & -b_1^{-2} - \frac{1}{b_1^2+b_2^2} & \frac{1}{b_1^2+b_2^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{b_1^2+b_2^2} & -\frac{1}{b_1^2+b_2^2} - b_2^{-2} & b_2^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2^{-2} & -2b_2^{-2} & b_2^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots \end{pmatrix}.$$

No chance to use Fourier transform, sums, etc... unless we answer to the previous open problem.

Use of the resolvent

Theorem

For any b_1 and b_2 positive the spectrum of the operator A satisfies

$$\sigma(A) \subset I = [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]. \tag{10}$$

For any $\omega \in I$ define

$$R^{\pm}(\omega) = \lim_{\epsilon \downarrow 0} R(\omega \pm i\epsilon).$$

We can prove that

$$R^{-}(\omega) = \overline{R}^{+}(\omega), \quad \forall \omega \in I.$$

Then

$$e^{itA} = \frac{1}{2i\pi} \int_I e^{it\omega}[R^{+}(\omega) - R^{-}(\omega)]d\omega$$

Big Problem: computing the resolvent

Lemma

Let $\lambda \in \mathbb{C} \setminus [-4\max\{b_1^{-2}, b_2^{-2}\}, 0]$. *Any solution of the equation* $(A - \lambda I)f = g$ *is given by*

$$f(j) = \frac{-r_s^{|j|}}{b_2^{-2}(1-r_2) + b_1^{-2}(1-r_1)} \Big[\sum_{k\in I_2} r_2^{|k|} g(k) + \sum_{k\in I_1} r_1^{|k|} g(k) \Big] \qquad (11)$$

$$+ \frac{b_s^2}{r_s - r_s^{-1}} \sum_{k\in I_s} (r_s^{|j-k|} - r_s^{|j|+|k|}) g(k), \quad j \in I_s,$$

where r_s, $s \in \{1,2\}$ *is the unique solution with* $|r_s| < 1$ *of the equation*

$$r_s^2 - 2r_s + 1 = \lambda b_s^2 r_s.$$

A small part of the proof

Let assume $b_2 < b_1$ and take $I = [-4b_1^{-2}, 0]$. "Essentially" we have to prove that

$$| \int_I e^{it\omega} r_1(\omega)^j r_2(\omega)^k | \leq C|t|^{-1/3}$$

uniformly on j and k, where

$$r_s^2 - 2r_s + 1 = \omega b_s^2 r_s, s \in \{1, 2\}.$$

On I, $r_1 = e^{i\theta_1(\omega)}$ and $r_2 = e^{i\theta_2(\omega)}$ and we have to prove that

$$| \int_I e^{it\omega} e^{ij\theta_1(\omega)} e^{ik\theta_2(\omega)} d\omega | \leq C|t|^{-1/3}$$

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$$|\int_I e^{it\omega} e^{ij\theta_1(\omega)} e^{ik\theta_2(\omega)} d\omega| \leq C|t|^{-1/3}$$

With a change of variables $\omega = 2b_1^{-2}(\cos\theta - 1)$ it remains to prove the following result

Lemma

Let $a \in (0,1]$. There exists a positive constant $C(a)$ such that the following

$$\Big| \int_0^\pi e^{it(2\cos\theta + 2z\arcsin(a\sin\frac{\theta}{2}))} e^{ity\theta} \sin\theta d\theta \Big| \leq C(a)(|t|+1)^{-1/3} \qquad (12)$$

holds for any real numbers y, z and t.

Obs: For $z = 0$ the estimate appears in the case of simpler DLSE.

Oscillatory integrals

Lemma (Van der Corput)

Suppose ψ *is real-valued and smooth in* I, *and that* $|\psi^{(k)}(x)| \geq 1$ *for all* $x \in I$. *Then*

$$\left| \int_I e^{i\lambda\psi(x)} \phi(x) dx \right| \leq c_k \lambda^{-1/k} (\|\phi\|_{L^\infty(I)} + \int_I |\phi'|).$$

We need to use two or three derivatives of the phase function

$$\psi_a(\theta) = 2\cos\theta + y\theta + z \arcsin(a \sin\frac{\theta}{2}).$$

But there are cases when the above Lemma is not sufficient

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Refinements of Van der Corput's Lemma

Lemma (Kenig, Ponce, Vega 91)

The following

$$
\begin{aligned}
& \left| \int_a^b e^{i(t\psi(\xi)-x\xi)} |\psi''(\xi)|^{1/2} \phi(\xi) d\xi \right| \\
& \qquad \le c_\psi |t|^{-1/2} \{ \|\phi\|_{L^\infty(a,b)} + \int_a^b |\phi'(\xi)| d\xi \}.
\end{aligned}
$$

holds for all real numbers x and t.

But there are cases when the above Lemma is still not sufficient

Refinements of Van der Corput's Lemma

Lemma (Kenig, Ponce, Vega 91)

The following

$$|\int_a^b e^{i(t\psi(\xi)-x\xi)}|\psi''(\xi)|^{1/2}\phi(\xi)d\xi|$$
$$\leq c_\psi|t|^{-1/2}\{\|\phi\|_{L^\infty(a,b)}+\int_a^b|\phi'(\xi)|d\xi\}.$$

holds for all real numbers x *and* t.

But there are cases when the above Lemma is still not sufficient

A new Lemma

Lemma (D. Stan, LI, JFAA 2011)

Assuming that at the critical points we have

$$\phi'(\xi) \sim \xi^{\alpha}, \alpha \geq 2$$

then

$$I(x,t) = \Big| \int_{\Omega} e^{i(t\phi(\xi) - x\xi)} |\phi'''(\xi)|^{\frac{1}{3}} d\xi \Big| \leq c t^{-\frac{1}{3}}.$$

Finally apply careful Van der Corput and KpV with $k = 2$ or $k = 3$ and even brute force

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Some Open Problems/Comments

I. Give sufficient conditions for a symmetric matrix A with few diagonals such that for the equation $iU_t + AU = 0$ we can prove similar decay properties, even with other type of decay: $t^{-1/4}$, etc.. (Work in progress by E/ Paraicu)

II. Coupling more than two equations ($\simeq$ included in C. Gavrus master thesis/SNSB)

III. Discrete potentials, etc...

Outline

Schrodinger equation on trees (or network trees)

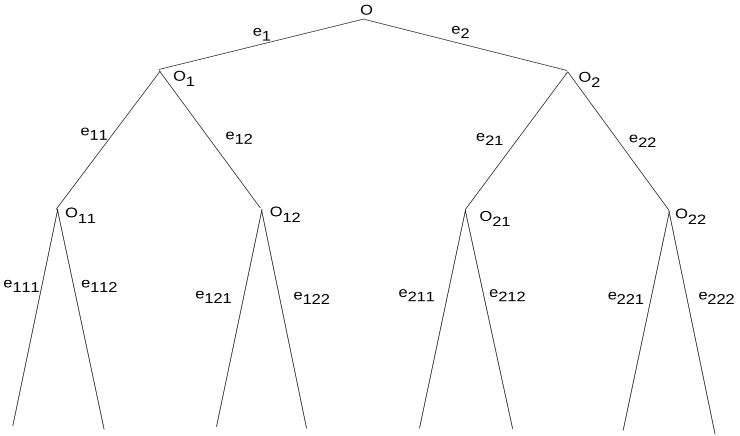
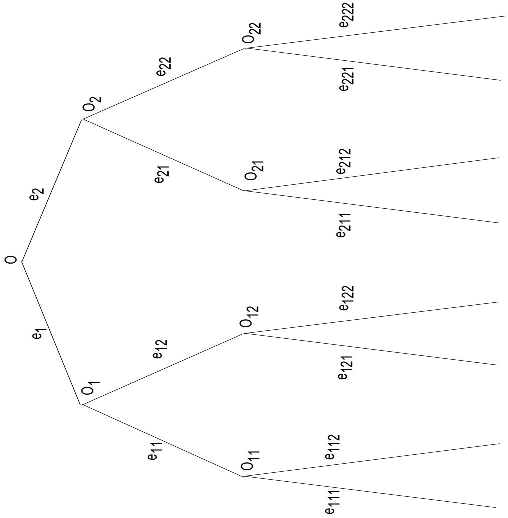


Figure: A tree with the third generation formed by infinite edges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O_{21}
O_{22}
e_{111}
e_{112}
e_{121}
e_{122}
e_{211}
e_{212}
e_{221}
e_{222}

$$\begin{cases} i\mathbf{u}_t(t,x) + \Delta_\Gamma \mathbf{u}(t,x) = 0, & x \in \Gamma, t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & x \in \Gamma. \end{cases} \tag{13}$$

$$\begin{cases} iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x \in (0,1), 1 \le |\overline{\alpha}| \le n, \\ iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x \in (0,\infty), |\overline{\alpha}| = n+1, \\ \begin{cases} u^{\overline{\alpha}}(t,1) = u^{\overline{\alpha\beta}}(t,0), & \beta \in \{1,2\}, 1 \le |\overline{\alpha}| \le n, \\ u^1(0,t) = u^2(0,t), \end{cases} \\ \begin{cases} u_x^{\overline{\alpha}}(t,1) = \sum_{\beta=1}^{2} u_x^{\overline{\alpha\beta}}(t,0), & 1 \le |\overline{\alpha}| \le n, \\ u_x^1(0,t) + u_x^2(0,t) = 0, \end{cases} \\ u^{\overline{\alpha}}(0,x) = u_0^{\overline{\alpha}}(x). \end{cases} \tag{14}$$

$$\begin{cases} i\mathbf{u}_t(t,x) + \Delta_\Gamma \mathbf{u}(t,x) = 0, & x \in \Gamma, t \neq 0, \\ \mathbf{u}(0) = \mathbf{u}_0, & x \in \Gamma. \end{cases} \tag{13}$$

$$\begin{cases} iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x \in (0,1), 1 \leq |\overline{\alpha}| \leq n, \\ iu_t^{\overline{\alpha}}(t,x) + u_{xx}^{\overline{\alpha}}(t,x) = 0, & x \in (0,\infty), |\overline{\alpha}| = n+1, \\ \begin{cases} u^{\overline{\alpha}}(t,1) = u^{\overline{\alpha\beta}}(t,0), & \beta \in \{1,2\}, 1 \leq |\overline{\alpha}| \leq n, \\ u^1(0,t) = u^2(0,t), \end{cases} \\ \begin{cases} u_x^{\overline{\alpha}}(t,1) = \sum_{\beta=1}^{2} u_x^{\overline{\alpha\beta}}(t,0), & 1 \leq |\overline{\alpha}| \leq n, \\ u_x^1(0,t) + u_x^2(0,t) = 0, \end{cases} \\ u^{\overline{\alpha}}(0,x) = u_0^{\overline{\alpha}}(x). \end{cases} \tag{14}$$

For regular trees we have similar dispersive estimates.
Main Tool: A result on LSE with discontinuous coefficients

Theorem (Banica, SIAM JMA 2003)

Consider a partition of the real axis $-\infty = x_0 < x_1 < \dots < x_{n+1} = \infty$ *and a step function* $\sigma(x) = \sigma_i$ *for* $x \in (x_i, x_{i+1})$, *where* σ_i *are positive numbers.*
The solution u *of the Schrödinger equation*

$$\begin{cases} iu_t(t,x) + (\sigma(x)u_x)_x(t,x) = 0, & \text{for } x \in \mathbb{R}, t \neq 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

satisfies the dispersion inequality

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C|t|^{-1/2}\|u_0\|_{L^1(\mathbb{R})}, \qquad t \neq 0.$$

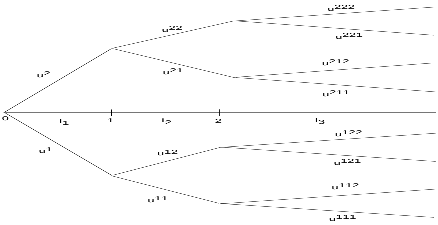
The star shaped tree

$$\begin{cases} iu^j_t(t,x) + u^j_{xx}(t,x) = 0, \quad x \in (0,\infty), 1 \le j \le n, \\ u^1(t,0) = u^2(t,0) = \cdots = u^n(t,0) \\ u^1_x(t,0) + u^2_x(t,0) + \cdots + u^n_x(t,0) = 0, \\ u^j(0,x) = u^j_0(x). \end{cases} \tag{15}$$

Making sums we can reduce the problem to the case of the half-line + Dirichlet or Neumann boundary conditions.

Idea of the proof in the case of regular trees

Look to the tree in a different way

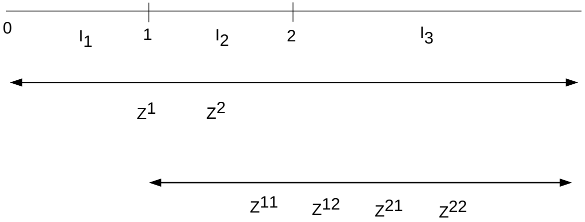


The functions situated above each interval are defined on that interval, for example u^1 and u^2 are defined on I_1, etc... where

$$I_k = \begin{cases} (k-1,k) & \text{if} \quad 1 \leq k \leq n, \\ (n,\infty) & \text{if} \quad k = n+1. \end{cases}$$

In order to obtain $L^1 - L^\infty$ estimates we need to introduce some averages

$$Z^{\overline{\alpha}} = \frac{\sum_\beta u^{\overline{\alpha\beta}}}{2^{|\beta|}} \quad \text{on} \quad I_{|\alpha|+|\beta|}, \quad 0 \le |\beta| \le n+1-|\alpha|$$



The first generation of Z's

$Z(t,x) = (-Z^1(t,-x), Z^2(t,x))$ satisfies

$$\begin{cases} iZ_t + Z_{xx} = 0 & x \in \mathbb{R}\backslash\{k, 1 \leq |k| \leq n\} \\ Z(t,k-) = Z(t,k+), & 1 \leq |k| \leq n \\ Z_x(t,k-) = 2Z_x(t,k+), & 1 \leq |k| \leq n \\ Z(0,x) = Z_0(x), & x \in \mathbb{R}\backslash\{k, 1 \leq |k| \leq n\}. \end{cases} \tag{16}$$

Using that Z satisfies

$$\|Z(t)\|_{L^\infty(\mathbb{R})} \leq |t|^{-1/2} \|Z(0)\|_{L^1(\mathbb{R})}$$

we have the same information about u^1 and u^2:

$$\max\{\|u^1(t)\|_{L^\infty(I_1)}, \|u^2(t)\|_{L^\infty(I_1)}\} \leq |t|^{-1/2} \sum_{k=1}^{n+1} \frac{1}{2^{k-1}} \Big\| \sum_{|\alpha|=k} u_0^{\overline{\alpha}} \Big\|_{L^1(I_k)}.$$

Next generations: induction
Question: What about a general tree? another ideas ...

The general case

Theorem (V. Banica, L.I, JMP 2011)

The solution of the linear Schrödinger equation on a tree is of the form

$$e^{it\Delta_\Gamma}u_0(x) = \sum_{\lambda\in\mathbb{R}} \frac{a_\lambda}{\sqrt{|t|}} \int_{I_\lambda} e^{i\frac{\phi_\lambda(x,y)}{t}} \, u_0(y)\,dy. \tag{17}$$

with $\phi_\lambda(x,y)\in\mathbb{R}$, $I_\lambda\in\{I_e\}_{e\in E}$, $\sum_{\lambda\in\mathbb{R}}|a_\lambda|<\infty$, *and it satisfies the dispersion inequality*

$$\|e^{it\Delta_\Gamma}u_0\|_{L^\infty(\Gamma)} \leq \frac{C}{\sqrt{|t|}}\|u_0\|_{L^1(\Gamma)}, \quad t\neq 0. \tag{18}$$

Ingredients for the proof

1. If $R_\omega \mathbf{f} = (-\Delta_\Gamma + \omega^2 I)^{-1}\mathbf{f}$ then $\omega R_\omega \mathbf{f}(x)$ can be analytically continued in a region containing the imaginary axis
2. A spectral calculus argument to write

$$e^{it\Delta_\Gamma}\mathbf{u}_0(x) = \int_{-\infty}^{\infty} e^{it\tau^2}\tau R_{i\tau}\mathbf{u}_0(x)\frac{d\tau}{\pi}.$$

3. The representation of the resolvent

$$\tau R_{i\tau}\mathbf{u}_0(x) = \sum_{\lambda\in\mathbb{R}} b_\lambda e^{i\tau\psi_\lambda(x)} \int_{I_\lambda} \mathbf{u}_0(y) e^{i\tau\beta_\lambda y} dy, \qquad (19)$$

with $\psi_\lambda(x), \beta_\lambda \in \mathbb{R}$, $I_\lambda \in \{I_e\}_{e\in E}$ and $\sum_{\beta\in\mathbb{R}} |b_\lambda| < \infty$.

Main steps

1.On each edge parametrized by I_e,

$$R_\omega \mathbf{f}(x) = ce^{\omega x} + \tilde{c}e^{-\omega x} + \frac{1}{2\omega}\int_{I_e} \mathbf{f}(y)\, e^{-\omega|x-y|}dy,\ \ x \in I_e.$$

2. The continuity of $R_\omega \mathbf{f}$ and of transmission of $\partial_x R_\omega \mathbf{f}$ at the vertices of the tree give the system of equations on the coefficients c's

3.

$$R_\omega \mathbf{f}(x) = \frac{1}{\omega \ \det D_\Gamma(\omega)} \sum_{\lambda=1}^{N(\Gamma)} c_\lambda e^{\pm\omega \Phi_\lambda(x)} \int_{I_\lambda} \mathbf{f}(y)\, e^{\pm \omega y} dy \tag{20}$$

$$+ \frac{1}{2\omega}\int_{I_e} \mathbf{f}(y)\, e^{-\omega|x-y|}dy, \tag{21}$$

4. Induction on the number of the vertices to prove that

$$\exists c_\Gamma, \epsilon_\Gamma > 0, \ |\det D_\Gamma(\omega)| > c_\Gamma, \forall \omega \in \mathbb{C}, |\Re \omega| < \epsilon_\Gamma.$$

5. Results on almost periodic functions to write

$$\frac{1}{\det D_\Gamma(i\tau)} = \sum_\lambda d_\lambda e^{i\tau\lambda}$$

with $\sum_\lambda |d_\lambda| < \infty$

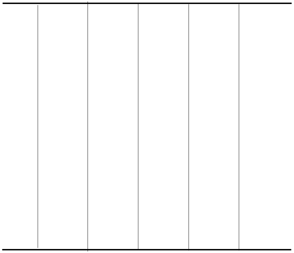
Some Open Problems/Comments

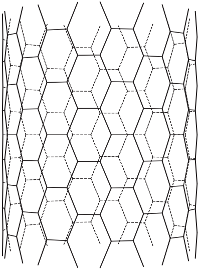
1. Other coupling conditions $A(v)\mathbf{f}(v) + B(v)\mathbf{f}'(v) = 0$ where
   1. the joint matrix $(A(v), B(v))$ has maximal rank, i.e. $d(v)$,
   2. $A(v)B(v)^T = B(v)A(v)^T$.

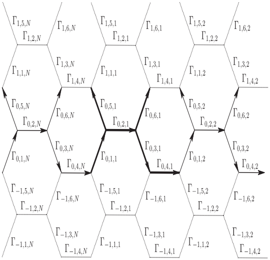
   LI and Banica, Analysis of PDE 2014, δ-coupling: $\sum (u_j)_x = \delta u$, $-\Delta + \sum_{k=1}^{N} \alpha_k \delta(x - x_k)$
2. clarify if the dispersion is possible only on trees or there are graphs (with some of the edges infinite) with suitable couplings where the dispersion is still true - some work in progress by A. Grecu
3. Some applications to control/stabilization on trees/networks
4. Discrete Schrödinger equations on trees, graphs - C. Gavrus
5. some magnetic operators: in the presence of an external magnetic field the effect of the topology of the graph becomes more pronounced
6. Strichartz estimates for “exotic” graphs
7. LSE with BV coefficients: N. Beli, with a lot of analytic number theory, multivariable polynomials, ODE, etc...

Exotic structures

- Dirac equation $iu_t = \mathcal{H}u + f(u)$ where $\mathcal{H} = \begin{bmatrix} -i\partial_x & -1 \\ -1 & i\partial_x \end{bmatrix}$







THANKS for your attention !!!