Optimal matching problems for the Euclidean distance

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An optimal matching problem

We are interested in an optimal matching problem that consists in transporting two commodities (say nuts and screws, we assume that we have the same total number of nuts and screws) to a prescribed location (say factories (to be located !) where we ensemble the nuts and the screws) in such a way that

1) they match (each factory receive the same number of nuts and of screws)

2) the total cost of the operation, measured in terms of the Euclidean distance that the commodities are transported, is minimized.

Monge-Kantorovich's Mass Transport Theory

Given $\mu, \nu \in \mathcal{M}^+(\Omega)$ satisfying the mass balance condition

$$\mu(\Omega) = \nu(\Omega).$$

The Monge problem. *Find T^* with $T^* \# \mu = \nu$ that minimizes the cost functional*

$$\tilde{\mathcal{F}}(T) := \int_\Omega |x - T(x)| \, d\mu(x)$$

A minimizer is called an optimal transport map of μ to ν.

Minimizers exist if $\mu = f\,dx$.

Monge-Kantorovich's Mass Transport Theory

In 1942, L. V. Kantorovich proposed to study a relaxed version of the Monge problem.

The Monge-Kantorovich problem. *The Monge-Kantorovich problem is the minimization problem*

$$\int_{\Omega\times\Omega} |x-y|\, d\gamma^*(x,y) = \min\left\{ \int_{\Omega\times\Omega} |x-y|\, d\gamma(x,y) \ : \ \gamma \in \Pi(\mu,\nu) \right\},$$

where

$$\Pi(\mu,\nu) := \{\textit{Radon measures } \gamma \textit{ in } \Omega\times\Omega : \pi_0\#\gamma = \mu, \pi_1\#\gamma = \nu\}.$$

A minimizer γ^* is an optimal transport plan. These minimizers always exist.

Monge-Kantorovich's Mass Transport Theory

The Monge-Kantorovich problem has a dual formulation.

Kantorovich-Rubinstein Theorem.

$$\min \left\{ \int_{\Omega\times\Omega} |x-y| \, d\gamma(x,y) \ : \ \gamma \in \Pi(\mu,\nu) \right\}$$
$$= \sup \left\{ \int_{\Omega} u \, d(\mu-\nu) \ : \ u \in K_1(X) \right\},$$

where $K_1(\Omega)$ is the set of 1-Lipschitz functions in Ω.

The maximizers u^* are called *Kantorovich potentials*.

The optimal matching problem

To write our optimal matching problem in mathematical terms, we fix two non-negative compactly supported functions $f^+,\ f^- \in L^\infty$, with supports $X_+,\ X_-$, respectively, satisfying the mass balance condition

$$M_0 := \int_{X_+} f^+ = \int_{X_-} f^-.$$

We also consider a compact set D (the target set).

The optimal matching problem

Then we take a large bounded domain Ω such that it contains all the relevant sets, the supports of f_+ and f_-, X_+, X_- and the target set D.

For simplicity we will assume that Ω is a convex $C^{1,1}$ bounded open set such that

$$X_+ \cap X_- = \emptyset, \quad (X_+ \cup X_-) \cap D = \emptyset \quad \text{and} \quad (X_+ \cup X_-) \cup D \subset\subset \Omega.$$

The optimal matching problem

For Borel functions $T_\pm : \Omega \to \Omega$ such that $T_+\#f^+ = T_-\#f^-$, we consider the functional

$$\mathcal{F}(T_+, T_-) := \int_\Omega |x - T_+(x)| f^+(x) dx + \int_\Omega |y - T_-(y)| f^-(y) dy.$$

The optimal matching problem

The optimal matching problem can be stated as the minimization problem

$$\min_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}(T_+, T_-),$$

where

$$\mathcal{A}_D(f^+, f^-) := \Big\{(T_+, T_-) : \ T_\pm : \Omega \to \Omega \ : T_\pm(X_\pm) \subset D, \\ \int_{T_+^{-1}(E)} f^+ = \int_{T_-^{-1}(E)} f^-, \ \forall E\Big\}.$$

The optimal matching problem

If $(T_+^*, T_-^*) \in \mathcal{A}_D(f^+, f^-)$ is a minimizer of the optimal matching problem, we will call the measure $\mu^* := T_+^* \# f^+ = T_-^* \# f^-$ a *matching measure* to the problem.

Note that there is no reason why a matching measure should be absolutely continuous with respect to the Lebesgue measure. In fact there are examples of matching measures that are singular.

The optimal matching problem

We have the following existence theorem.

Theorem *The optimal matching problem has a solution, that is, there exist Borel functions* $(T_+^*, T_-^*) \in \mathcal{A}_D(f^+, f^-)$ *such that*

$$\mathcal{F}(T_+^*, T_-^*) = \inf_{(T_+, T_-) \in \mathcal{A}_D(f^+, f^-)} \mathcal{F}(T_+, T_-).$$

Moreover, we can obtain a solution $(\tilde{T}_+, \tilde{T}_-)$ *of the optimal matching problem with a matching measure supported on the boundary of D.*

The optimal matching problem

We have two different proofs to this existence theorem.

The first one is more direct but does not provide a constructive way of getting the optimal matching measure in D, which is one of the unknowns in this problem. The main tool is the use of ingredients from the classical Monge-Kantorovich theory.

The second proof is by approximation of the associated Kantorovich potentials by a system of $p-$Laplacian type problems when p goes to ∞.

$p-$Laplacian results

Let us consider the following variational problem

$$\min_{\substack{(v,w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \\ v \le w \text{ in } D}} \frac{1}{p}\int_\Omega |Dv|^p + \frac{1}{p}\int_\Omega |Dw|^p - \int_\Omega vf^+ + \int_\Omega wf^-.$$

Theorem *There exists a minimizer (v_p, w_p). In addition any two minimizers differ by a constant, that is, if (v_p, w_p) and $(\tilde{v}_p, \tilde{w}_p)$ are minimizers then there exists a constant c with $v_p = \tilde{v}_p + c$ and $w_p = \tilde{w}_p + c$.*

$p-$Laplacian results

Now we observe that we can pass to the limit as $p \to \infty$ in a subsequence of minimizers.

Theorem *Up to a subsequence,*

$$\lim_{p\to\infty} (v_p, w_p) = (v_\infty, w_\infty) \quad \textit{uniformly,}$$

where (v_∞, w_∞) *is a solution of the variational problem*

$$\max_{\substack{v,\, w \,\in\, W^{1,\infty}(\Omega) \\ |\nabla v|_\infty,\, |\nabla w|_\infty \,\le\, 1 \\ v \,\le\, w \text{ in } D}} \int_\Omega v f^+ - w f^- .$$

$p-$Laplacian results

Remark The limit (v_∞, w_∞) gives a pair of Kantorovich potentials for our optimal matching problem.

But in fact this limit procedure gives much more since it allows us to identify the optimal matching measure

$p-$Laplacian results

lemma There exists a positive Radon measure h_p of mass M_0 such that

1. $$\begin{cases} -\Delta_p v_p = f^+ - h_p & \text{in } \Omega, \\ |\nabla v_p|^{p-2}\nabla v_p \cdot \eta = 0 & \text{on } \partial\Omega, \end{cases}$$

   $$\begin{cases} -\Delta_p w_p = h_p - f^- & \text{in } \Omega, \\ |\nabla w_p|^{p-2}\nabla w_p \cdot \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

2. The positive measure h_p is supported on $\{x \in D : v_p(x) = w_p(x)\}$.

$p-$Laplacian results

lemma Up to a subsequence,

$$h_p \rightharpoonup h_\infty \quad \text{as } p \to \infty, \text{ weakly}^* \text{ as measures},$$

with h_∞ a positive Radon measure of mass M_0 supported on $\{x \in D : v_\infty(x) = w_\infty(x)\}$. And the limit (v_∞, w_∞) satisfies:

v_∞ is a Kantorovich potential for the transport of f^+ to h_∞,

w_∞ is a Kantorovich potential for the transport of h_∞ to f^-,

with respect to the Euclidean distance.

$p-$Laplacian results

Therefore, we have obtained the following theorem.

Theorem *The measure* h_∞ *is a matching measure to the optimal matching problem.*

In some cases (D being the closure of a smooth domain) this approximation procedure selects a matching measure supported on the boundary of the target set.

Delta measures

Let us characterize the set of configurations for which the matching measure is a delta concentrated at a point $z_0 \in D$.

Theorem *Assume that there is a point $z_0 \in D$ such that for any pair of points $x \in X_+$ and $y \in X_-$ we have*

$$\min_{z \in D}\{|x - z| + |y - z|\} = |x - z_0| + |y - z_0|,$$

then the measure $M_0 \delta_{z_0}$ is an optimal matching measure. The converse also holds.

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THANKS !!!!.