Blow-up for a non-local equation

Raúl Ferreira

Universidad Complutense de Madrid

INTERNATIONAL CENTER FOR ADVANCED STUDIES

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- Then the rate at which individuals are arriving to position x from any other place is given

$$\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy = (J*u)(x,t)$$

- Rate at which they are leaving location x to travel to any other site,

$$\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy = u(x,t)$$

Simple Nonlocal Model

$$u_t = J * u - u = \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))\, dy$$

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However there are important differences between them. For instance, non-local equation has no regularizing effect.

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In fact, if we consider $J_\varepsilon(x) = \frac{C_J}{\varepsilon^{N+2}} J(\frac{x}{\varepsilon})$, the solutions $u_\varepsilon \to u$ which is a solution of the heat equation

Nonlocal Neumann Model

- We can impose that the individuals only jump in a certain domain,

$$\begin{cases} u_t = \displaystyle\int_\Omega J(x-y)(u(y,t)-u(x,t))\,dy & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

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- If we consider J_ε the solution converges to the solution of

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial u}{\partial \nu} = 0 & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

- Assuming that individuals jumping outside of Ω die.

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Nonlocal Dirichlet Model

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$$\begin{cases} u_t = \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))\,dy & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

- In this case, with J_ε we converge to a solution of

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times \mathbb{R}^+ \\ u = 0 & \text{in } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

- A nonlocal version of the the p-laplacian operator

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$$

is given by

$$\Delta_p^J u = \int_{\mathbb{R}^N} J(x-y)|u(y,t) - u(x,t|^{p-2}(u(y,t) - u(x,t))\, dy$$

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- If we rescale the kernel in the Cauchy, Dirichlet and Neumann problems then we converge to the corresponding local problems.

- For the local p-laplacian it is well known that there exists $\lambda_1 > 0$

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  and the corresponding eigenfunction function $\phi_1 > 0$.
- However for the non-local version

$$\lambda_1 = \inf_{\substack{u \in L^p(\Omega) \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega}} \frac{\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^p \, dxdy}{\int_\Omega u^p \, dx}$$

  we only know that $\lambda_1 > 0$. The existence of the corresponding positive eigenfunction has not been shown, due to the lack of compactness of the minimizer sequence.

Our problem

We analyze some features of the blow-up phenomenon arising from the non-local diffusion problem,

$$\begin{cases} u_t(x,t) = \Delta_p^J u + u^q & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

- $q \geq 0$, $p \geq 2$
- Ω is a bounded smooth domain
- $J : \mathbb{R}^N \to \mathbb{R}$, $J \in C_0^\infty(\mathbb{R}^N)$ is a nonnegative, bounded and symmetric kernel ($J(z) = J(-z)$), such that $\int_{\mathbb{R}^N} J(z)dz = 1$.
- $u_0(x)$, continuous, positive initial datum.

We can rewrite the equation as

$$\begin{aligned} u(x,t) = \;& u_0(x) + \int_0^t \int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy\,dt \\ & - \int_0^t |u(x,t)|^{p-2}u(x,t) \int_{\mathbb{R}^N\setminus\Omega} J(x-y)\,dy\,dt + \int_0^t u^q(x,t)\,dt \end{aligned}$$

Existence and uniqueness

$$\mathfrak{D}(u) = u_0(x) + \int_0^t \int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy\,dt - \int_0^t |u(x,t)|^{p-2}u(x,t)\int_{\mathbb{R}^N\setminus\Omega} J(x-y)\,dy\,dt + \int_0^t u^q(x,t)\,dt$$

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In fact, $\mathfrak{D} : X_{t_0} \to X_{t_0}$, and for small t_0 it is a strict contraction in a ball of X_{t_0}.

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Therefore, if $T < \infty$ it holds that

$$\limsup_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty(\overline{\Omega})} = +\infty.$$

The energy functional

The energy functional associated to our problem is

$$
\begin{aligned}
H(v(t)) &= \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|v(y,t)-v(x,t)|^p dx\,dy \\
&\quad -\frac{1}{q+1}\int_\Omega |v|^q v(x,t)\,dx + \frac{1}{p}\int_\Omega\int_{\mathbb{R}^N\setminus\Omega} J(x-y)|v|^{p-1}v(x,t)\,dy\,dx.
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Multiplying the equation by u_t, it holds

$$\frac{d}{dt}H(u(t)) = -\int_\Omega (u_t)^2(x,t)\,dx.$$

Thus H is non-increasing along the orbits.

Lemma

If $q+1 \geq p$ and $\exists t_0$ such that $H(u(t_0)) < 0$. Then u blows up in finite time.

Proof. Multiplying the equation by u, it holds

$$L'(u(t)) = -pH(u(t)) + \left(1 - \frac{p}{q+1}\right) \int_\Omega u^{q+1}\, dx.$$

where

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Assuming that $q+1 > p \geq 2$

$$L'(u(t)) \geq C\left(L(u(t))\right)^{\frac{q+1}{2}}$$

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On the other hand

$$(L'(u(t)))^2 = \left(\frac{d}{dt}\left(\frac{1}{2}\int_\Omega u^2 dx\right)\right)^2 = \left(\int_\Omega u u_t dx\right)^2 \leq \int_\Omega u^2 dx \int_\Omega (u_t)^2 dx$$

Blow-up criterium

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$$(L'(u(t)))^2 \leq -2L(u(t))\frac{d}{dt}H(u(t)) = \frac{2}{p}L(u(t))L''(u(t))$$

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□

In both cases

$$\left(\int_\Omega u^2(x)\,dx\right)^{\frac{1}{2}} \le C(T-t)^{\frac{-1}{q+1}}.$$

Lemma

If $q+1<p$ and u_0 satisfies

$$\int_{\mathbb{R}^N} J(x-y)|u_0(y)-u_0(x)|^{p-2}(u_0(y)-u_0(x))\,dy + u_0^q(x) \geq 0,$$

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Proof. Assume that $\exists\, t_n \to T$ such that $\lambda_n = \|u(\cdot,t_n)\|_\infty \to \infty$ and scale the solutions as follows

$$\omega_n(x,\tau) = \lambda_n^{-1} u(x, \lambda_n^{2-p}\,\tau + t_n), \quad \text{for} \quad \tau \in \left[-\frac{\lambda_n^{p-2} t_n}{2}, 0\right] := I_n$$

Global existence criterium

Since $u_t \geq 0$ then $(\omega_n)_\tau \geq 0$ and

$$0 \leq \frac{u_0(x)}{\lambda_n} \leq \omega_n(x,\tau) \leq 1 = \|\omega_n(\cdot,0)\|_\infty = \omega(x_n,0)$$

for all $(x,\tau) \in \overline{\Omega} \times I_n$ and some $x_n \in \overline{\Omega}$.

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for all $(x,\tau) \in \overline{\Omega} \times I_n$ and some $x_n \in \overline{\Omega}$.

There exists a subsequence n_k such that w_{n_k} converge in L^∞-weak* to ω_∞, defined in $\overline{\Omega} \times (-\infty, 0]$ and satisfying

$$0 \leq \omega_\infty \leq 1, \quad \omega_\infty(x_\infty, 0) = 1,$$

where x_∞ is the limit of x_{n_k}.

The scaled functions verify

$$(\omega_n)_\tau(x,\tau) = \int_\Omega J(x-y)|\omega_n(y,\tau)-\omega_n(x,\tau)|^{p-2}(\omega_n(y,\tau)-\omega_n(x,\tau))\,dy$$
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In particular, at $(x,\tau)=(x_n,0)$

$$\int_\Omega J(x_n-y)|\omega_n(y,0)-1|^{p-1} \le \lambda_n^{q-p+1} - \int_{\mathbb{R}^N\setminus\Omega} J(x_n-y)\,dy.$$

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By convexity of $f(s)=|s-1|^{p-1}$ and Fatou's Lemma

$$\int_\Omega J(x_\infty-y)|\omega_\infty(y,0)-1|^{p-1}\,dy \le -\int_{\mathbb{R}^N\setminus\Omega} J(x_\infty-y)\,dy\,.$$

Global existence criterium

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In terms of the function u it reads as

$$\liminf_{t_{n_k}\to T} \frac{u(x,t_{n_k})}{\|u(\cdot,t_{n_k})\|_\infty} = 1\,, \qquad x \in B_1(x_\infty).$$

Therefore, $u(x,t)$ blows up a.e. in $B_1(x_\infty)$

Global existence criterium

On the other hand, we can rewrite the functional energy as

$$H(u(t)) = \frac{1}{2p}\int_\Omega\int_{\mathbb{R}^N} J(x-y)|u(y,t)-u(x,t)|^p dx\, dy - \frac{1}{q+1}\int_\Omega u^{q+1}(x,t)\, dx$$

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\end{aligned}
$$

Then, using the following Poincaré inequality

There exists $\lambda(J,\Omega,p) > 0$ such that

$$
\lambda \int_{\Omega} |u(x)|^p dx \leq \int_{\Omega} \int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^p dy\, dx,
$$

for all $u \in L^p(\Omega)$ with $u = 0$ outside of Ω

Global existence criterium

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Since $p > q + 1$ we get

$$H(u(t)) \to \infty$$

□

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  reads as

  $$\rho^q - \rho^{p-1}\int_{\mathbb{R}^N\setminus\Omega} J(x-y)\,dy \geq 0\,.$$

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In the general case, for $A > 1$ the function

$$w(x,t) = Au_1(x, A^{p-2}t)\,,$$

is a global supersolution of our problem.

Note that the energy functional for u_0 takes the form

$$H(u_\rho) = \frac{\rho^p \mu(\Omega)}{p} \int_{\mathbb{R}^N \setminus \Omega} J(x-y)\, dy - \frac{\rho^{q+1}\mu(\Omega)}{q+1}$$

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- Case $q+1 > p$, $u_\rho(x,t)$ blows up for $\rho \geq \left(\frac{q+1}{p}\right)^{\frac{1}{q+1-p}}$.
  If Ω is a *thin domain*, that is, for all $x \in \bar{\Omega}$,

$$\int_{\mathbb{R}^N \setminus \Omega} J(x-y)dy \geq \beta > 0,$$

then small constant functions are stationary supersolutions. Thus, we also have global solution in this range.

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Theorem

Let $q+1 \geq p > 2$ and u be a solution that blows up at time T. Then

$$\max_{x\in\overline{\Omega}} u(x,t) \geq C_q (T-t)^{-\frac{1}{q-1}}, \qquad C_q = \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}.$$

Theorem

Let $q+1>p\geq 2$ and u be a solution blowing up at time T. Then, there exists a positive constant such that

$$\max_{x\in\Omega} u(x,t)\leq C(T-t)^{-\frac{1}{q-1}}\,.$$

Proof. Adding and subtracting u^{p-1}, our equation can be written as follows

$$\begin{aligned} u_t(x,t) = \quad & \int_{\mathbb{R}^N} J(x-y)(|u(x,t)|^{p-2}u(x,t)-|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t)))\,dy \\ & +u^q(x,t)-u^{p-1}(x,t)\,, \end{aligned}$$

Theorem

Let $q+1>p\geq 2$ and u be a solution blowing up at time T. Then, there exists a positive constant such that

$$\max_{x\in\Omega} u(x,t) \leq C(T-t)^{-\frac{1}{q-1}} .$$

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Since the function $f(s)=|s|^{p-2}s$ is increasing, the integral term is positive.

Therefore

$$u_t(x,t) \geq u^q(x,t) - u^{p-1}(x,t)$$

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By integration

$$u(x,t) \leq C(T-t)^{\frac{-1}{q+1}}$$

□

Theorem

Let $q+1=p$, $2<p<3$ and u be a solution which blows up at time T. Then there exists a positive constant C such that

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Proof. We rescale the solutions as follows

$$\omega(x,s)=(T-t)^{\frac{1}{p-2}}u(x,t), \qquad s=-\log\left(\frac{T}{T-t}\right),$$

Let $x^* \in \Omega$ such that $\omega(x^*(s),s)=\max_\Omega \omega(\cdot,s)$

At this point the equation for ω reads

$$\omega_s(x^*,s) = \int_{\mathbb{R}^N} J(x-y)\left[|\omega(x^*,s)|^{p-2}\omega(x^*,s) - |\omega(x^*,s)-\omega(y,s)|^{p-2}(\omega(x^*,s)-\omega(y,s))\right] dy, \\ -\frac{1}{p-2}\omega(x^*,s)$$

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We apply the Mean Value Theorem to the function $f(z) = |z|^{p-2}z$ to obtain

$$\omega_s(x^*,s) = -\frac{1}{p-2}\omega(x^*,s) + (p-1)\int_{\mathbb{R}^N} J(x^*-y)|\xi|^{p-2}\omega(y,s)\,dy$$

for some $\omega(x^*,s)-\omega(y,s) \le \xi \le \omega(x^*,s)$.

$$\omega_s(x^*,s) \le -\frac{1}{p-2}\omega(x^*,s) + (p-1)\omega^{p-2}(x^*,s)\int_{\mathbb{R}^N} J(x^*-y)\omega(y,s)\,dy$$

Blow-up rates

$$\omega_s(x^*, s) \leq \left(C\omega^{p-3}(x^*, s) - \frac{1}{p-2} \right) \omega(x^*, s).$$

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Since $p < 3$, if ω is large it is increasing. So, ω is bounded. □

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Let $q \geq p$ and $\Omega = B_R(0)$. If $u(x,t)$ is radially symmetric and decreasing, then $B(u) = \{0\}$.

Blow-up sets

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Gap for $p-1 < q < p$

As before, we rescale the solutions as follows

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Then, ω is bounded and it satisfies

$$\begin{aligned}\omega_s(x,s) = \quad & e^{-\frac{q+1-p}{q-1}s} \int_{\mathbb{R}^N} J(x-y)|\omega(y,s)-\omega(x,s)|^{p-2}(\omega(y,s)-\omega(x,s))\,dy \\ & +\omega^q(x,s) - \tfrac{1}{p-2}\omega(x,s).\end{aligned}$$

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- $q+1>p$.

$$\lim_{s\to\infty}\omega(x,s) = \begin{cases} C \\ 0 \end{cases}$$

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$$+\omega^q(x,s) - \tfrac{1}{p-2}\omega(x,s).$$

- $q+1>p$. If $\omega(x,s)\to 0$

$$\omega_s(x,s) \le Ce^{-\frac{q+1-p}{q-1}s} - \frac{1-\varepsilon}{p-2}\omega(x,s)$$

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$$\omega_s(x,s) \le Ce^{-\frac{q+1-p}{q-1}s} - \frac{1-\varepsilon}{p-2}\omega(x,s)$$

Then,

$$u(x,t) = e^{\frac{1}{q-1}s}\omega(x,s) \le \begin{cases} C & q \ge p \\ C\,e^{\frac{p-q}{q-1}s} & q < p \end{cases}$$

The Neumann problem

Theorem

- If $q > 1$ any solution to (2) blows up, while if $q \leq 1$ every solution is global.
- Rates:
  - If $1 < q$ then $\max_{\overline{\Omega}} u(\cdot, t) \geq C_q (T - t)^{-\frac{1}{q-1}}$.
  - If $1 < q$ and $q \neq p - 1$, then $\max_{\overline{\Omega}} u(\cdot, t) \leq C (T - t)^{-\frac{1}{q-1}}$.
  - If $q = p - 1$ and, either $2 < p < 3$ or Ω is a thin domain, then $\max_{\overline{\Omega}} u(\cdot, t) \leq C (T - t)^{-\frac{1}{q-1}}$ (*).
- Sets : the flat solution $z(t) = C_q (T - t)^{-\frac{1}{q-1}}$ blows up globally. However, there exists also single-point blow-up.
  - If $1 < q < p - 1$ then $B(U) = \overline{\Omega}$.
  - Assuming (*). If $1 < q = p - 1$,then $B(U) = \overline{\Omega}$.
  - If $q > p$ then we have single-point blow-up.