CIMPA LECTURES 2019

"Generalised geometry and supersymmetric spaces"

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1. MOTIVATION, G-STRUCTURES

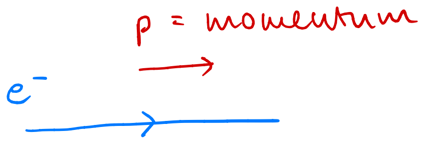
- What is the idea behind these lectures? String theory is a quantum theory of gravity. Classically we have

$$\text{gravity} = \text{Riemannian geometry} \qquad (\text{Einstein})$$
$$\text{symmetries} = \text{diffeomorphisms}$$

Quantum consistency suggests string theory has "huge" set of symmetries vastly extending diffeomorphisms.

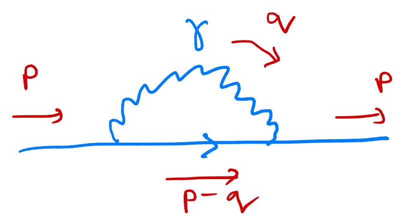
Why? because there are lots of "miraculous" cancellations

ASIDE: mass of the electron: "Feynman diagrams"



classical

$\text{mass} = m_0$



quantum correction

$\Delta\text{mass} = ?$

naively $q = (E, \vec{q})$

$$\Delta\text{mass} = \int_0^\Lambda dE\,(\ \ ) \sim \Lambda \qquad \text{dimensional analysis}$$

actually

$$\Delta\text{mass} = m_0 \int_0^\Lambda \frac{dE}{E}\,(\ \ ) \sim m_0 \ln\Lambda \qquad \text{"softer"}$$

why? because extra "chiral symmetry" when
$m_0 = 0$ (separate left- and right-hand electrons)
Similar cancellations for every Feyn. diagram in
string theory.

Some questions (we will not answer them all!)

- what kind of geometry/symmetries underlie string theory?
- Is Riemannian geometry the only way to define a geometrical theory of gravity?
- does string theory define natural extensions of complex, symplectic, hyper-Kähler, Calabi-Yau ... geometries?

Also more practically:

- compactification: 4d universe $\times M$
  - $\rightsquigarrow$ what kind of internal geometries M give supersymmetric 4d theories
  - $\rightsquigarrow$ how does 4d theory depend on M? moduli?

(see Mariana Graña's lectures)

- AdS - cft correspondence (Maldacena)

$AdS_{D+1} \times M \longleftrightarrow$ conformal field theory in D-dim.

c.f. hyperbolic space

gravity — field theory

differential geom. — algebraic geom.

⇝ what are allowed geoms. of M?

⇝ how related to algebraic description?

1.1 G structures

- Let's do Riemannian geometry in a slightly unconventional way.

M : d-dim manifold with coordinates x^m

TM : tangent space

$\Gamma(TM) \ni v = v^m(x)\, \partial/\partial x^m$: vector field

- Differential structure is encoded in $v, w \in \Gamma(TM)$

Lie bracket: $[v,w] = (v^n \partial_n w^m - w^n \partial_n v^m)\, \partial/\partial x^m$
$\in \Gamma(TM)$

More generally given $v \in \Gamma(TM)$ and a tensor:

$$\alpha = \alpha^{m_1 \cdots m_p}{}_{n_1 \cdots n_q}\, \partial/\partial x^{m_1} \otimes \cdots \otimes \partial/\partial x^{m_p} \otimes dx^{n_1} \otimes \cdots \otimes dx^{n_q}$$

$$\in \Gamma(TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M)$$

Lie derivative: = infinitesimal diffeomorphism.

$\partial_\rho = \partial / \partial x^\rho$

$$(\mathcal{L}_v \alpha)^{m_1 \cdots m_p}{}_{n_1 \cdots n_q} = v^\rho \partial_\rho \alpha^{m_1 \cdots m_p}{}_{n_1 \cdots n_q}$$
$$- (\partial_\rho v^{m_1}) \alpha^{\rho m_2 \cdots m_p}{}_{n_1 \cdots n_q} - \cdots - (\partial_\rho v^{m_p}) \alpha^{m_1 \rho \cdots m_p}{}_{n_1 \cdots n_q}$$
$$+ (\partial_{n_1} v^\rho) \alpha^{m_1 \cdots m_p}{}_{\rho n_2 \cdots n_q} + \cdots + (\partial_{n_q} v^\rho) \alpha^{m_1 \cdots m_p}{}_{n_1 \cdots n_{q-1} \rho}$$
$$= v^\rho \partial_\rho \alpha^{m_1 \cdots m_p}{}_{n_1 \cdots n_p} - (\partial x_{ad} v) \cdot \alpha^{m_1 \cdots m_p}{}_{n_1 \cdots n_q}$$

$\partial_\rho v^a$ matrix $\in gl(n, \mathbb{R})$

so that $\mathcal{L}_v w = [v, w]$ if $\alpha = w \in \Gamma(TM)$

- One can define a frame at a point $x \in M$

$$\{ \hat{e}_a = \hat{e}_a{}^m \partial / \partial x^m \} \in T_x M \quad \text{form a basis}$$

so that we can expand $T_x M \ni v = v^a \hat{e}_a$.

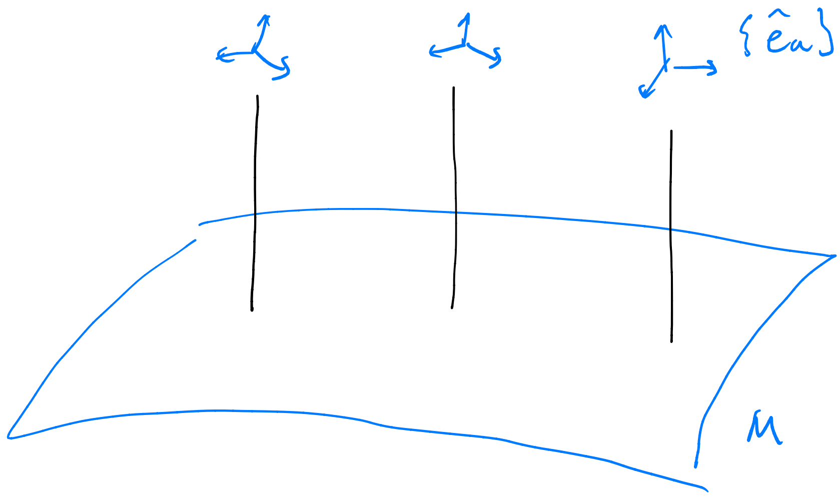
$$F_x = \text{set of frames at } x \in M$$

Any 2 frames are related by

$$\hat{e}'_a = M_a{}^b \hat{e}_b \qquad M \in GL(d, \mathbb{R})$$

then

$$\text{frame bundle } FM = \coprod_{x \in M} F_x$$



on a patch $U \subset M$ local basis

$\{\hat{e}^a(x)\} \in \Gamma(FU)$ such that $v(x) = v^a(x)\hat{e}_a(x)$

Geometrical

FM is a "principal G-bundle"

= bundle where fibres form a group G

with $G = GL(d, \mathbb{R})$.

- What is a G-structure, P? Given $G \subset GL(d,\mathbb{R})$:

Def: P is a principal G sub-bundle $P \subset FM$

This just means we have a subset of frames $P_x \subset F_x$ at each point such that

if $\{\hat{e}_a\}, \{\hat{e}'_a\} \in P_x$ then $\hat{e}'_a = M_a{}^b e_b$

where $M \in G \subset GL(d, \mathbb{R})$

- example 1: $O(d)$-structure $\Leftrightarrow$ metric, g

$\Leftarrow$ Suppose we have a metric $g \in \Gamma(T^*M \otimes T^*M)$

$$|v|^2 = g(v,v) = g_{mn} v^m v^n$$

and define

$$P_x = \text{set of orthonormal frames}$$
$$= \{ \{\hat{e}_a\} \in F_x : \ g(\hat{e}_a, \hat{e}_b) = \delta_{ab} \}$$

then $M \in O(d)$ (preserves δ_{ab})

P is an $O(d)$-structure

$\Rightarrow$ given a frame $\{\hat{e}_a\} \in P_x$, define dual frame

$$e^a(\hat{e}_b) = e^a{}_m \hat{e}^m{}_b = \delta^a{}_b \qquad e^a \in T^*_x M$$

then

$$g_{mn} = \delta_{ab} e^a{}_m e^b{}_n$$

- example 2 : $GL(n, \mathbb{C}) \Leftrightarrow$ "almost complex structure I"

$$I : TM \to TM \qquad \text{such that} \qquad I^2 = -\mathbb{1}$$

$$v^m \mapsto I^m{}_n v^n$$

structure underlying complex manifolds $(d = 2n)$

- example 3 : $Sp(2n, \mathbb{R}) \Leftrightarrow$ "almost symplectic struc ω"

$$\omega \in \Gamma(\Lambda^2 T^* M) \qquad \omega_{mn} = -\omega_{nm} \qquad \text{non-degenerate}$$

structure underlying symplectic manifolds. $(d = 2n)$

etc. etc....

- Usually the case

$$\text{G-structure } P \Leftrightarrow \text{invariant tensors } \Xi :$$
$$(\text{eg } g, I, \omega, \dots)$$

The existence of P depends on the topology of the manifold – restricts the way the tangent bundle can twist.

$O(d)$ structure – always exists.

$SL(d, \mathbb{R})$ structure – orientible manifold

etc...