Exercises: Generalised geometry and supersymmetric spaces

G-structures and intrinsic torsion

1. What invariant tensor defines an $SL(d,\mathbb{R}) \subset GL(d,\mathbb{R})$ structure? How does the existence of such a structure constrain the topology of the manifold M?

   Suppose you have two $SL(d,\mathbb{R})$ compatible connections ∇ and ∇' and define $\nabla' - \nabla = \Omega$. Show that Ω is a tensor. How is it constrained?

2. Show that the torsion-free condition for an almost complex structure, defined by I and given by
$$[v,w] \in \Gamma(T^{1,0}) \qquad \forall v,w \in \Gamma(T^{1,0}), \tag{1}$$
is equivalent to the vanishing of the Nijenhuis tensor
$$N_{np}^{m} := I_q^m \left(\partial_n I_p^q - \partial_p I_n^q\right) - \left(I_n^q \partial_q I_p^m - I_p^q \partial_q I_n^m\right) = 0. \tag{2}$$
(Note that you can use I to project onto $T^{1,0}$.)

3. Consider a "product structure", that is a map $R : TM \to TM$ such that $R^2 = \mathbf{1}$. Show that R defines a $GL(p,\mathbb{R}) \times GL(q,\mathbb{R}) \subset GL(n,\mathbb{R})$ structure where $n = p+q$ and hence gives a decomposition of the tangent space as
$$TM \simeq L_1 \oplus L_2 \tag{3}$$
where L_1 and L_2 have dimension p and q respectively.

   Show that vanishing of the intrinsic torsion is equivalent to the pair of conditions
$$\begin{aligned} [v,w] &\in \Gamma(L_1) \qquad \forall v,w \in \Gamma(L_1), \\ [v,w] &\in \Gamma(L_2) \qquad \forall v,w \in \Gamma(L_2). \end{aligned} \tag{4}$$

4. Consider the case were M is a Lie group with a basis for left-invariant vectors fields $\{l_a\}$ satisfying
$$[l_a, l_b] = f_{ab}{}^c l_c, \tag{5}$$
where $f_{ab}{}^c$ are the structure constants of the Lie algebra. Argue that the l_a define an "indentity structure" or "parallelisation", that is a G-structure with a trivial group $G = \{\mathbf{1}\}$.

   What is the intrinsic torsion?

5. Consider an almost symplectic structure defined by a two-form $\omega \in \Gamma(\Lambda^2 T^* M)$. Show that the $Sp(2n,\mathbb{R})$ structure is torsion-free if and only if $\mathrm{d}\omega = 0$.