Basics on Calabi-Yau manifolds $\boxed{1}$

§1. Smooth vs complex manifolds

SMOOTH MANIFOLD

Let M be a set. A smooth (real) manifold structure of dim $D\in\mathbb{N}$ (on M) is a collection $\{(U, x_U)\}_{U\in\mathcal{U}}$ (atlas, charts), with $U\subseteq M$, $\mathcal{U}$ countable, $\bigcup_{U\in\mathcal{U}} U = M$, $x_U : U \xrightarrow[\text{bijection}]{\sim} x_U(U) \underset{\text{open}}{\subseteq} \mathbb{R}^D$,

$$x_V\circ x_U^{-1}\big|_{x_U(U\cap V)} : x_U(U\cap V)\longrightarrow x_V(U\cap V)$$

is a C^∞-diffeomorphism, $\forall\, U,V\in\mathcal{U}$. We also impose that given $p,q\in M$, $p\neq q$, there exist $U'\subseteq U$, $V'\subseteq V$ s.th. $U'\cap V'=\emptyset$, $x_U(U')$, $x_V(V')$ open.

<u>Example</u>/<u>Exercise</u>: Let $S^2=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ be the 2-sphere. Endow it with the charts (U_+,x_+) and (U_-,x_-) given by $U_+=S^2\setminus\{(0,0,1)\}$, $U_-=S^2\setminus\{(0,0,-1)\}$, $x_\pm : U_\pm\to\mathbb{R}^2$ defined as $x_\pm(x,y,z)=\Big(\underbrace{\frac{x}{1\mp z}}_{x_\pm}, \underbrace{\frac{y}{1\mp z}}_{y_\pm}\Big)$. Verify that this defines a smooth structure of dim 2 on S^2.

$\boxed{\text{Rk}}$ We should consider the notion of "equivalence" of atlases.

A smooth map $f: M\longrightarrow N$ is a map s.th., for every $p\in M$, there is a chart (U,x_U) at p of M, a chart (V,y_V) at $f(p)$ of N s.th. $f(U)\subseteq V$ and $y_V\circ f\circ x_U^{-1}: x_U(U)\longrightarrow y_V(V)$ is smooth.

COMPLEX

A complex manifold of complex dimension $d\in\mathbb{N}$ is given by the previous def. by replacing "$\mathbb{R}^D$" by "$\mathbb{C}^d$", and "C^∞-diffeo" by "biholomorphic". The analogous change gives the definition of holomorphic map between complex manifolds.

$\boxed{\text{Fact}}$ By identifying $\mathbb{C}^d\cong\mathbb{R}^{2d}$ by $(x_1+iy_1,\dots,x_d+iy_d)\mapsto(x_1,\dots,x_d,y_1,\dots,y_d)$ and the fact that a holomorphic map $\mathbb{C}^d\xrightarrow{g}\mathbb{C}^{d'}$ is the same as a smooth map $\mathbb{R}^{2d}\xrightarrow{g}\mathbb{R}^{2d'}$ s.th. $j_{d'}\circ Dg(p)=Dg(p)\circ j_d$, where $j_d := \begin{pmatrix} 0 & -I_d \\ I_d & 0\end{pmatrix}$ is the matrix of the multiplication by i, we see that complex manifold $\Longrightarrow$ smooth manifold. $\square$

Example/Exercise (the 2-sphere, cont'd) Define $Z_\pm : U_\pm \to \mathbb{C}$ by $Z_\pm(p) = X_\pm(p) + iY_\pm(p)$ for $p \in U_\pm$. Prove that $\{(U_+, Z_+), (U_-, Z_-)\}$ gives a structure of complex manifold.
Hint: Prove that $Z_- = 1/Z_+$ in $U_+ \cap U_-$.

§2. Interlude in vector bundles (optional)

$\boxed{\text{rank of the v.b}}$

VECTOR BUNDLE

A vector bundle with fibre (a vector space of fin dim r) F over a smooth manifold M is a smooth manifold E and a smooth surjective map $\pi : E \to M$ s.th. there is a collection $\{(U, \underset{\text{trivialization}}{\varphi_U})\}_{U \in \mathcal{U}}$, where $\bigcup_{U \in \mathcal{U}} U = M$, U open in M,

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi_U", "\sim"' , "\text{bijection}"'] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "pr"] \\ & U \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

and

$$\varphi_V \circ \varphi_U^{-1}\big|_{(U\cap V)\times F} : (U \cap V) \times F \longrightarrow (U \cap V) \times F$$
$$(p, v) \longmapsto (p, \underbrace{(\varphi_V \circ \varphi_U^{-1})_p}_{F \to F \text{ linear iso}}(v))$$

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A map $M \xrightarrow{s} E$ s.th. $\pi \circ s = \mathrm{id}_M$ is a **section**. $\boxed{\text{Notation}}$: $E_p = \pi^{-1}(\{p\})$. □

Example/Exercise (the tangent bundle) Let M be a smooth manifold of dim D. For $p \in M$, set $T_pM = \{d : \underset{\text{smooth maps from } M \text{ to } \mathbb{R}}{C^\infty(M, \mathbb{R})} \to \mathbb{R} \mid d \text{ is } \mathbb{R}\text{-linear},\ d(fg) = f(p)d(g) + d(f)g(p)\}$. Then T_pM is a $\mathbb{R}$-v.s. of dim d. If (U, x_U) is a chart at p, then $\left\{ \underbrace{\frac{\partial}{\partial x_U^i}\Big|_p}_{\text{also denoted } \partial_i} \right\}_{i=1,\dots,D}$ $\left(= f \mapsto \frac{\partial(f \circ x_U^{-1})}{\partial x^i}\Big|_{x_U(p)} \right)$ is a basis of T_pM. Define $TM = \bigsqcup_{p \in M} T_pM$. Prove it is a v.b. of rank D.

Operations on vector bundles:

- **dual**: $E \xrightarrow{\pi} M$ v.b. with fibre F $\Rightarrow$ $E^* \xrightarrow{\pi} M$ v.b. with fibre F^*
- **direct sum**: $E \xrightarrow{\pi} M$, $E' \xrightarrow{\pi'} M'$ v.b with fibres F, F', resp $\Rightarrow$ $E \oplus E' \to M$ v.b. with fibre $F \oplus F'$
- **tensor product**: $E \xrightarrow{\pi} M$, $E' \xrightarrow{\pi'} M'$ v.b. with fibres F, F', resp $\Rightarrow$ $E \otimes E' \xrightarrow{\pi} M$ v.b. with fibre $F \otimes F'$
- **pull-back**: $E \xrightarrow{\pi} M$, v.b. with fibre F, $f : N \to M$ smooth $\Rightarrow$ $f^*E = \{(e, q) \mid \pi(e) = f(q)\} \to N$ v.b with fibre F

Example (cotangent bundle): The dual $(TM)^*$ of TM is denoted by T^*M and it is called the cotangent bundle. The dual basis of $\left\{ \frac{\partial}{\partial x_U^i}\Big|_p \right\}_{i=1,\dots,D}$ (for a chart (U, x_U)) is denoted by $\{dx_U^i|_p\}_{i=1,\dots,D}$, i.e. $\left\langle dx_U^i|_p, \frac{\partial}{\partial x_U^j}\Big|_p \right\rangle = \delta_{ij}$.

V.B. MOR

A morphism $f : E \to E'$ of v.b. over M is a smooth map s.th.
$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$
and $f|_{E_p} : E_p \to E'_p$ is linear, $\forall p \in M$.

§3. Almost complex manifolds

An almost complex structure on a d-dim smooth manifold M is a v.b. morphism $J : TM \to TM$ s.th. $J \circ J = -I_{TM}$. Notation: $J_p = J|_{T_pM}$.

Prop If M is almost complex, then its dimension is even.

(Proof) $\det(J_p)^2 = \det(J_p^2) = \det(-Id_{T_pM}) = (-1)^{\dim T_pM} = (-1)^{\dim M}$ □

Fact A complex manifold M has a canonical almost complex structure where J_p = multiplication by i $(= j_{\dim M})$.

Exercise: Let J be an almost complex structure on a manifold M of dim $D = 2d$. Prove that, for $p \in M$, there is a basis of T_pM s.th. $[J_p] = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$. (*)

Prop An almost complex manifold is orientable (i.e. there is an atlas $\{(U, x_U)\}_{U \in \mathcal{U}}$ s.th. $\det D(x_V \circ x_U^{-1})(p) > 0, \forall p \in x_U(U \cap V), \forall U, V \in \mathcal{U}$).

Exercise: Prove it! Hint: Use charts $\{(U, x_U)\}$ s.th. J is given by (*) w.r.t. $\left\{\frac{\partial}{\partial x_U^i}\right\}$.

⚠ Not all even dim smooth manifolds are almost complex. For instance, S^4 has no almost complex structure. Moreover, Borel and Serre proved that the only even dim spheres with an almost complex structure are S^2 and S^6.

We still don't know if S^6 has complex structure (G. Etesi articles, M. Atiyah's articles)

§4. Towards Hodge theory

Let M be almost complex with map $J : TM \to TM$.
Set $TM_{\mathbb{C}} = \bigsqcup_{p \in M} T_pM \otimes_{\mathbb{R}} \mathbb{C}$ and let $J : TM_{\mathbb{C}} \to TM_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of the previous J.

Then: (1) J has eigenvalues i and $-i$
(2) J is diagonalizable and the complex dim of the eigenspaces coincide.

Call the eigenspace of i (resp. $-i$) $T_p^{1,0}M$ (resp. $T_p^{0,1}M$):

$T_p^{1,0}M = \{X \in T_pM_{\mathbb{C}} \mid JX = iX\}$ & $T_p^{0,1}M = \{X \in T_pM_{\mathbb{C}} \mid JX = -iX\}$

Then $T^{1,0}M$ and $T^{0,1}M$ are v.b. over M and

$$T^{1,0}M = \{X - iJX \mid X \in TM\}, \quad T^{0,1}M = \{X + iJX \mid X \in TM\}$$

(holomorphic vector fields; anti-holomorphic vector fields)

$\boxed{\text{For a complex manifold}}$ If (U, z_U) is a complex chart $z_U : U \to \mathbb{C}^d \simeq \mathbb{R}^{2d}$ consider $x_U, y_U : U \to \mathbb{R}^d$ s.th. $z_U = x_U + i y_U$. Then $J\left(\frac{\partial}{\partial x_U^j}\right) = \frac{\partial}{\partial y_U^j}$.
Set $\frac{\partial}{\partial z_U^j} := \frac{1}{2}\left(\frac{\partial}{\partial x_U^j} - i\frac{\partial}{\partial y_U^j}\right) \in T^{1,0}M$, $\frac{\partial}{\partial \bar{z}_U^j} := \frac{1}{2}\left(\frac{\partial}{\partial x_U^j} + i\frac{\partial}{\partial y_U^j}\right) \in T^{0,1}M$. They give bases!

Let $\Lambda^\bullet M = \bigoplus_{k=0}^{\dim M} \Lambda^k(T^*M)$ and $\Lambda^\bullet_{\mathbb{C}} M = \Lambda^\bullet M \otimes_{\mathbb{R}} \mathbb{C}$. Set

$$\left.\begin{aligned} \Lambda^{1,0}M &= \{\xi \in \Lambda^1_{\mathbb{C}} M \mid \langle \xi, Z\rangle = 0, \forall Z \in T^{0,1}M\} \\ \Lambda^{0,1}M &= \{\xi \in \Lambda^1_{\mathbb{C}} M \mid \langle \xi, Z\rangle = 0, \forall Z \in T^{1,0}M\} \end{aligned}\right\} \text{— vector subbundles of } \Lambda^1_{\mathbb{C}}M$$

Moreover, set

$$\Lambda^{p,q}M = (\Lambda^{1,0}M)^{\otimes p} \otimes (\Lambda^{0,1}M)^{\otimes q}$$

Note that
$\Lambda^{1,0}M = \{\omega - i\omega\circ J \mid \omega \in \Lambda^1 M\}$
$\Lambda^{0,1}M = \{\omega + i\omega\circ J \mid \omega \in \Lambda^1 M\}$

Then

$$\Lambda^k_{\mathbb{C}} M = \bigoplus_{p+q=k} \Lambda^{p,q}M$$

sections of $\Lambda^{p,q}M = \Omega^{p,q}M$

$\boxed{\text{For a complex manifold}}$ If (U, z_U) is a complex chart $z_U : U \to \mathbb{C}^d \simeq \mathbb{R}^{2d}$ and let x_U, y_U as before. Then set $dz_U^j = dx_U^j + i\, dy_U^j \in \Lambda^{1,0}M$, $d\bar{z}_U^j = dx_U^j - i\, dy_U^j \in \Lambda^{0,1}M$.
They form bases! Also, $\left\{ dz_U^{j_1}\big|_p \wedge \dots \wedge dz_U^{j_p}\big|_p \wedge d\bar{z}_U^{k_1}\big|_p \wedge \dots \wedge d\bar{z}_U^{k_q}\big|_p \;\middle|\; \begin{matrix} j_1 < \dots < j_p \\ k_1 < \dots < k_q \end{matrix} \right\}$
is a basis of $\Lambda^{p,q}_p M$. (check it!)

Remember that there is a map $d : \Lambda^k M \to \Lambda^{k+1}M$ sending $f(x)\, dx_U^{j_1} \wedge \dots \wedge dx_U^{j_k}$ to $\sum_{j=1}^{D} \frac{\partial f}{\partial x_U^j} dx_U^j \wedge dx_U^{j_1} \wedge \dots \wedge dx_U^{j_k}$. Denote its complex extension $\Lambda^k_{\mathbb{C}} M \to \Lambda^{k+1}_{\mathbb{C}} M$ by d. Then $d(\Lambda^{p,q}M) \subseteq \Lambda^{p-1,q+2}M \oplus \Lambda^{p,q+1}M \oplus \Lambda^{p+1,q}M \oplus \Lambda^{p+2,q-1}M$ (check it)

Denote the component $\Lambda^{p,q}M \to \Lambda^{p+1,q}M$ by ∂ and $\Lambda^{p,q}M \to \Lambda^{p,q+1}M$ by $\bar{\partial}$.

$\boxed{\text{THM}}$ (Newlander-Nirenberg) Let (M,J) be almost complex. TFAE

(1) (M,J) comes from a complex manifold structure

(2) The bracket of two (anti-)holomorphic vector fields is (anti-)holomorphic.

(3) $d = \partial + \bar{\partial}$

(4) $N^J = 0$, where $N^J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \forall X,Y \in \Gamma(TM)$.

Recall that the bracket $[X,Y]$ of $X = \sum_{j=1}^{D} X^j \frac{\partial}{\partial x_U^j}$, $Y = \sum_{j=1}^{D} Y^j \frac{\partial}{\partial x_U^j}$ is $\sum_{j,i}\left(X^j \frac{\partial Y^i}{\partial x^j}\frac{\partial}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i}\right)$. □

Exercise: If M is a complex manifold, show that $d \circ d = 0$ implies that $\partial \circ \partial = \bar{\partial} \circ \bar{\partial} = \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$

Define $H^{p,q}_{\bar{\partial}}(M) = \mathrm{Ker}\left(\bar{\partial}\big|_{\Lambda^{p,q}M}\right) \Big/ \mathrm{Im}\left(\bar{\partial}\big|_{\Lambda^{p,q-1}M}\right)$ ← Dolbeault cohomology

and $h^{p,q} = \dim H^{p,q}_{\bar{\partial}}(M)$

§5. Entering Riemannian geometry

RIEMANNIAN

Recall that any smooth manifold M has a Riemannian structure, i.e. a section $g \in \Gamma((Sym_2(TM))^*)$ s.th. $g_p : T_pM \otimes_{\mathbb{R}} T_pM \to \mathbb{R}$ is nondegenerate, $\forall p \in M$.

Recall that a covariant derivative on a v.b. $E \xrightarrow{\pi} M$ is a map

$$\nabla : \Gamma(TM) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E) \qquad \equiv \qquad \Gamma(E) \xrightarrow{\nabla} \Gamma(E \otimes T^*M)$$

$$(X, Y) \longmapsto \nabla_X Y$$

s.th. $\begin{cases} \nabla_{fX} Y = f \cdot \nabla_X Y \\ \nabla_X (f \cdot Y) = f \nabla_X Y + df(X) \cdot Y \end{cases}$ $\longleftrightarrow$ s.th. $\nabla(f \cdot Y) = f \cdot \nabla Y + Y \otimes df$

$\boxed{\nabla_X Y = \langle \nabla Y, X \rangle}$ $\quad \forall f \in C^\infty(M),\ X \in \Gamma(TM),\ Y \in \Gamma(E)$

If $E = TM$, the torsion of ∇ is $T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$.

$\boxed{\text{Fact}}$: If M is Riemannian, there is a unique cov. der. ∇ on TM s.th. $T_\nabla \equiv 0$ and $\nabla(g) \equiv 0$, i.e. $d(g(Y,Z))(X) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

It is called the Levi-Civita covariant derivative $\longrightarrow$ $\boxed{\text{Christoffel symbols}}$ $\boxed{\nabla_{\partial_\alpha} \partial_\beta = \sum_\gamma \Gamma^\gamma_{\alpha\beta} \partial_\gamma}$

Exercise: If $g = (g_{\alpha\beta})$ in a chart, prove that $\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} \sum_\delta g^{\delta\gamma} (\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta})$.

Example/Exercise: Using the embedding $S^2 \subseteq \mathbb{R}^3$, identify T_pS^2 with the plane in $\mathbb{R}^3$ perpendicular to p. Define $g_p(X,Y) \stackrel{\text{def}}{=} \langle X, Y \rangle_{\mathbb{R}^3}$. Prove it is a Riemannian metric.

$\uparrow$ Euclidean scalar product

Compute g in the charts $(U_\pm, x_\pm)$: verify that

$$g_{U_\pm, p} = \frac{4}{(1 + \|x_\pm(p)\|^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example/Exercise: Using the previous formula compute $\Gamma^\gamma_{\alpha\beta}$ for S^2 in both charts $(U_\pm, x_\pm)$.

Verify that (in (U_+, x_+))

$$\left.\begin{aligned} \Gamma^1_{11} &= \frac{-2X_+}{1+X_+^2+Y_+^2}, & \Gamma^1_{12} &= \frac{-2Y_+}{1+X_+^2+Y_+^2}, & \Gamma^2_{12} &= \frac{-2X_+}{1+X_+^2+Y_+^2} \\ \Gamma^1_{22} &= \frac{2X_+}{1+X_+^2+Y_+^2}, & \Gamma^2_{11} &= \frac{-2Y_+}{1+X_+^2+Y_+^2}, & \Gamma^2_{22} &= \frac{-2Y_+}{1+X_+^2+Y_+^2} \end{aligned}\right\} \longrightarrow \begin{aligned} \nabla_{\partial_1}\partial_1 = -\nabla_{\partial_2}\partial_2 &= \frac{-2X_+\partial_1 - 2Y_+\partial_2}{1+X_+^2+Y_+^2} \\ \nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 &= \frac{-2X_+\partial_2 - 2Y_+\partial_1}{1+X_+^2+Y_+^2} \end{aligned}$$

CURVATURE

The curvature of a cov. der. ∇ on TM is $R(\nabla) : \Gamma(M)^{\otimes 3} \longrightarrow \Gamma(M)$ by $R(\nabla)(X,Y,Z) = \nabla_X \circ \nabla_Y (Z) - \nabla_Y \circ \nabla_X (Z) - \nabla_{[X,Y]}(Z)$. Equivalently, we will see $R(\nabla) \in \Gamma(End(TM) \otimes \Lambda^2 T^*M)$.

PARALLEL TRANS.

If $\gamma:[0,1]\to M$ is a piecewise smooth curve, ∇ a cov. der on v.b. $E \xrightarrow{\pi} M$, then $\gamma^*E \to [0,1]$ is a v.b. with induced cov. der $\gamma^*(\nabla)$. Given $v \in E_{\gamma(0)}$, there is a unique section $s \in \Gamma(\gamma^* E)$ s.th. $\nabla_{\dot\gamma(t)} s(t) = 0, \forall t \in [0,1]$, & $s(0) = v$. The map $P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ given by $s(0) \mapsto s(1)$ is the <u>parallel transport</u> of v.

Define, for ∇ the Levi-Civita cov. der: $\boxed{\text{holonomy group}}$

$$\mathrm{Hol}_p(\nabla) := \{P_\gamma : T_pM \to T_pM \mid \gamma \text{ loop s.th. } \gamma(0)=\gamma(1)=p\} \underset{\text{subgroup}}{\subseteq} O(T_pM) \subseteq GL(T_pM)$$

$$\mathrm{Hol}^0_p(\nabla) := \{P_\gamma : T_p \to T_pM \mid \gamma \text{ null-homotopic loop}, \gamma(0)=\gamma(1)=p\}$$

$\boxed{\text{restricted holonomy group}}$

$\boxed{\text{Prop}}$
- If M connected of dim D, $\mathrm{Hol}_p(\nabla)$ and $\mathrm{Hol}^0_p(\nabla)$ are independent of $p \in M$ (up to conj.)
- $\mathrm{Hol}^0_p(\nabla)$ is a Lie subgroup of $SO(D)$ [Yamabe], and it is closed [Borel-Lichnerowicz]
- $\mathrm{Hol}^0_p(\nabla)$ is the conn. comp. of $\mathrm{Hol}_p(\nabla)$ and normal: $\pi_1(M,p) \longrightarrow \mathrm{Hol}_p(\nabla)/\mathrm{Hol}^0_p(\nabla)$.

Define $\mathfrak{hol}_p(\nabla)$ to be the Lie algebra of $\mathrm{Hol}^0_p(\nabla)$. Note that $\mathfrak{hol}_p(\nabla) \subseteq \mathfrak{so}(D)$.

<u>Example</u>/<u>Exercise</u>: Compute $R(\nabla)$ for the previous example. Verify that $R(\partial_1,\partial_2,\partial_1) = \dfrac{-4\partial_2}{(1+x_+^2+y_+^2)^2}$, $R(\partial_1,\partial_2,\partial_2) = \dfrac{4\partial_1}{(1+x_+^2+y_+^2)^2} \longrightarrow R(\partial_1,\partial_2) = \dfrac{4}{(1+x_+^2+y_+^2)^2}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ in (U_+, x_+)

$\boxed{\text{THM}}$ (Ambrose–Singer) Let M be a Riemannian manifold, connected, ∇ the Levi-Civita cov. der. Then $R(\nabla)_p \in \mathfrak{hol}_p(\nabla) \otimes_{\mathbb{R}} \Lambda^2 T_p^*M$, and in fact $\mathfrak{hol}_p(\nabla)$ is the vector subspace of $\mathrm{End}(T_pM)$ spanned by $P_\gamma^{-1} \cdot (R(\nabla)_q(-,v,w)) P_\gamma$, $\forall \gamma:[0,1]\to M$, $\gamma(0)=p$, $\gamma(1)=q$, $v,w \in T_qM$!

<u>Example</u>/<u>Exercise</u>: The previous thm and the computation say that $\mathrm{Hol}_p(\nabla) = SO(2)$ for S^2.

§6. <u>Hermitian</u> geometry

HERMITIAN

A Riemannian metric g on a smooth manifold M provided with a (almost) complex str. is (almost) <u>Hermitian</u> if
$$g(JX, JY) = g(X,Y), \ \forall X, Y \in \Gamma(TM).$$

$\hookrightarrow$ Extend g_p to $T_pM_{\mathbb{C}} \otimes_{\mathbb{C}} T_pM_{\mathbb{C}} \to \mathbb{C}$ by $\mathbb{C}$-linearity

$\boxed{\text{Fact}}$ Any two (anti-)holomorphic vectorfields are orthogonal.

(Proof) $g(Z,W) = g(JZ,JW) = i^2 g(Z,W) = -g(Z,W)$ □

$\boxed{\text{Fact}}$ A complex manifold always has a Hermitian metric.

(Proof) For a Riemannian metric g, define $h(X,Y) = \frac{1}{2}(g(X,Y) + g(JX,JY))$. □

FUND. FORM [If (M,J,g) is a Hermitian manifold, set $\underbrace{\omega(X,Y)}_{\text{FUNDAMENTAL FORM}} = g(JX,Y)$, $\forall X,Y \in \Gamma(TM)$

Rk: $\omega(X,Y) = g(JX,Y) = g(J^2X,JY) = -g(X,JY) = -g(JY,X) = -\omega(Y,X)$.

$\boxed{\text{Fact}}$ $\omega \in \Lambda^{1,1}M$, it is nondegenerate, $\omega(JX,JY) = \omega(X,Y)$ $\forall X,Y \in \Gamma(TM)$ and $\omega_p(X,JX) > 0$. $(\forall X \neq 0)$

KÄHLER [A Hermitian manifold (M,J,g) is Kähler if $d\omega = 0$

$\boxed{\text{THM}}$ Let (M,J,g) be an almost Hermitian manifold of dim $2d$, let ∇ be the Levi-Civita covariant derivative. TFAE

(i) (M,J) is complex and g is Kähler

(ii) $\nabla J = 0$

(iii) $\nabla \omega = 0$

(iv) $\mathrm{Hol}(\nabla) \subseteq U(d)$ (and J is associated to the $U(d)$-structure)

Example/Exercise: On S^2, prove that the usual Riemannian metric g is Hermitian and

$$\omega = \frac{2\,dX_+ \wedge dY_+}{1 + X_+^2 + Y_+^2} = \frac{i}{(1+|Z_+|^2)}\, dZ_+ \wedge d\bar{Z}_+ \quad \text{in } (U_+, \varkappa_+)$$

§7. Hodge theory

Let M be Kähler of dim $2d$, compact $\longrightarrow$ it is orientable, it is Riemannian (with metric g)

$\rightsquigarrow$ volume form $dV_g = \frac{1}{d!}\underbrace{\omega \wedge \dots \wedge \omega}_{d \text{ factors}}$

$\rightsquigarrow * : \Lambda^k_{\mathbb{C}} M \longrightarrow \Lambda^{2d-k}_{\mathbb{C}} M$

Define
$$d^*\alpha = -*d(*\alpha)$$
$$\partial^*\alpha = -*\partial(*\alpha)$$
$$\bar{\partial}^*\alpha = -*\bar{\partial}(*\alpha)$$

Then, if $\boxed{\langle\,,\rangle = \int_M (\,,) dV_g}$ $\alpha \wedge (*\bar{\beta}) = (\alpha, \beta)\, dV_g$

$\longrightarrow \langle \alpha, d^*\beta\rangle = \langle d\alpha, \beta\rangle$ and also for $\partial, \bar{\partial}$

sesquilinear extension of $(\,,) : \Lambda^k_p M \otimes \Lambda^k_p M \to \mathbb{R}$ induced by g.

Set
$$\Delta_d = dd^* + d^*d \quad \text{and also } \Delta_\partial, \Delta_{\bar{\partial}}$$

Note that $*(\Lambda^{p,q}M) = \Lambda^{d-p,d-q}M$

$\boxed{\text{Fact}}$: $\Delta_\partial = \Delta_{\bar{\partial}} = \Delta_d / 2$

Set $\mathcal{H}^{p,q} = \mathrm{Ker}(\Delta|_{\Lambda^{p,q}M})$

$\boxed{\text{Prop}}$ $\Lambda^{p,q}M = \underbrace{\partial(\Lambda^{p,q-1}M) \oplus \overbrace{\mathcal{H}^{p,q}}^{}}_{} \oplus \bar{\partial}^*(\Lambda^{p,q+1}M)$, where $\overbrace{\mathcal{H}^{p,q} \oplus \bar{\partial}^*(\Lambda^{p,q+1}M)}^{\mathrm{Ker}(\bar{\partial}^*|_{\Lambda^{p,q}M})}$ and $\underbrace{\partial(\Lambda^{p,q-1}M) \oplus \mathcal{H}^{p,q}}_{\mathrm{Ker}\,\bar{\partial}|_{\Lambda^{p,q}M}}$. Hence $\mathcal{H}^{p,q} \cong H^{p,q}_{\bar{\partial}}(M)$

Moreover $H^k(M,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M)$,

$H^{p,q}_{\bar{\partial}}(M) \simeq \overline{H^{q,p}_{\bar{\partial}}(M)}$ and $H^{p,q}_{\bar{\partial}}(M) \simeq H^{d-p,d-q}_{\bar{\partial}}(M)$

§8. Calabi-Yau manifolds

RICCI TENSOR

For a Riemannian manifold M with Levi-Civita cov. der ∇ and curvature $R(\nabla)$, define $\widehat{Ric} \in \Gamma(T^*M^{\otimes 2} \otimes End(TM))$ by $\widehat{Ric}(X,Y,Z) = R(X,Z,Y)$ $\forall X,Y,Z \in \Gamma(TM)$ and the Ricci tensor $Ric \in \Gamma(T^*M^{\otimes 2})$ by $Ric_p(X,Y) = Tr \circ \widehat{Ric}_p(X,Y,-)$

A Riemannian manifold is Ricci-flat if $Ric \equiv 0$.

Define the Ricci form $\rho(X,Y) = Ric(JX,Y)$, if M is almost complex.

$\boxed{\text{Prop}}$ Let M be a Kähler manifold. Then $\rho \in \Lambda^{1,1}M \cap \Lambda^2 M$, $d\rho = 0$. Moreover $[\rho] \in H^2(M,\mathbb{R})$ equals $2\pi c_1(M)$

first Chern class of M (holomorphic information)

Remarkable result: Let M be Kähler of dim $2d$, ∇ be the Levi-Civita cov. der. $Hol^0(\nabla) \subseteq SU(d)$ iff M is Ricci-flat.

Example/Exercise: For S^2, prove that $\rho = \frac{2}{(1+|z_+|^2)^2} dz_+ \wedge d\bar{z}_+$ in (U_+, x_+)

$\boxed{\text{Def}}$ Let M be a compact Kähler manifold of dim $2d$. We say that M is Calabi-Yau if $Hol(\nabla) \subseteq SU(d)$.

$\boxed{\text{Prop}}$ For a compact Kähler manifold M of dim $2d$, t.f.a.e.

(1) $\Lambda^{d,0}M$ is the trivial line bundle (i.e. $M \times \mathbb{C}$)

(2) $\exists\, \omega \in \Gamma(\Lambda^{d,0}M)$, nowhere vanishing

(3) M is Calabi-Yau.

$\boxed{\nabla}$ A CY manifold is Ricci-flat but the converse does not hold in general (Enriques surfaces). It holds if M is simply connected (why?).

$\boxed{\text{THM}}$ (Calabi conjecture, Yau) Let (M,J) be a compact Kähler manifold with metric g. Let $\rho' \in \Lambda^{1,1}M \cap \Lambda^2 M \cap Ker(d)$ s.th. $[\rho'] = 2\pi c_1(M)$, and let ω be the Kähler form. Then, there exists a Kähler metric g' on M with Kähler form ω' s.th. $[\omega'] = [\omega]$ in $H^2(M,\mathbb{R})$ with Ricci form ρ'.

Counterexample: S^2 is not Calabi-Yau ∇

4' **Example** (the torus)

For $a, b \in \mathbb{R}$ define the complex curve $M = \{(u,v) \in \mathbb{C}^2 \mid v^2 = u^3 + a.u + b\}$ (with $4a^3 + 27b^2 \neq 0 \Rightarrow$ nonsingular ?)

This a complex manifold of complex dim 1 ?

It is automatically Kähler ? (why)

Prove that $\Omega = \frac{du}{v}$ gives a nowhere vanishing holomorphic 1-form $\Rightarrow$ M is CY.