String Compactifications

Exercise 1.1:

Find the explicit expression for the energy-momentum tensor for the Polyakov action using the definition

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_P}{\delta h^{\alpha\beta}}$$

and show it is traceless ($h^{\alpha\beta}T_{\alpha\beta} = 0$). Relate the trace $T^{\alpha}{}_{\alpha}$ to the variation of the action under a Weyl transformation $h_{\alpha\beta} \to e^{2\omega}h_{\alpha\beta}$. In D-dimensional flat space ($g_{\mu\nu} = \eta_{\mu\nu}$), relate the conservation law $\nabla^{\alpha}T_{\alpha\beta} = 0$ to the equations of motion for the fields X^{μ}.

Exercise 1.2:

Compute the mode expansion and mass spectrum of a closed string with periodic boundary conditions for the coordinates $X^0, ..., X^{24}$ and:

$$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi m R$$

where m is integer. When can we make sense of this solution? What condition does one get from the constraint $\int_0^{\pi} \dot{X} \cdot X' = 0$?

Exercise 1.3:

Show that the state

$$|\phi >= e_{\mu}\alpha^{\mu}_{-1}|0>$$

has positive norm for e^{μ} spacelike.

Exercise 1.4:

a) Find the mass squared of the following open-string states

$$|\phi_1 >= \alpha^{\mu}_{-1}|0>, \qquad |\phi_2 >= \alpha^{\mu}_{-1}\alpha^{\nu}_{-1}|0>$$

$$|\phi_3 >= \alpha^{\mu}_{-3}|0>, \qquad |\phi_4 >= \alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\alpha^{\rho}_{-2}|0>$$

b) Find the mass squared of the following closed-string states

$$|\phi_1 >= \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0>, \qquad |\phi_2 >= \alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\tilde{\alpha}^{\rho}_{-2}|0>$$

c) What can you say about the following closed-string state?

$$|\phi_3 > \alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-2}|0>$$

Exercise 2.1:

Consider a 26-dimensional space-time with metric

$$ds^2 = G_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu + e^{2\sigma}(dx^{25} + A_\mu dx^\mu)^2$$

where $x^{25} \sim x^{25} + 2\pi R$. Show that the 26-dimensional action

$$S_{26} = \frac{1}{16\pi G_N^{(26)}} \int d^{26}x\sqrt{-G}\left(R^{(26)} + 4\partial_\mu\phi\partial^\mu\phi\right)$$

after integrating over x^{25}, becomes

$$S_{25} = \frac{1}{16\pi G_N^{(25)}} \int d^{25}x\sqrt{-g}\left(R^{(25)} + F_{\mu\nu}F^{\mu\nu} + \partial_\mu\sigma_0\partial^\mu\sigma_0 + 4\partial_\mu\phi_0\partial^\mu\phi_0 + \text{KK tower}\right)$$

where σ_0 and ϕ_0 are the zero modes. Find the relation between the 26 and the 25-dimensional Newton constants.

Exercise 2.2:

Show that for the closed bosonic string compactified on a circle at the self-dual radius $R = \sqrt{\alpha'}$ there are 4 extra massless vectors and 8 extra massless scalars coming from states with non-zero momentum and/or winding number along the circle.

Exercise 2.3:

Show that for the closed bosonic string compactified on a torus T^k at at particular point in moduli space such that there is an enhancement of symmetry to a group $G \times G$, with G os dimension d and rank k, there are a total of d^2 massless scalars (on top of the zero mode of the dilaton).

Exercise 2.4:

Consider T^2 compactifications parametrized by the coordinates x^8 and x^9. Define the complex moduli ρ by $\rho = B_{89} + i\text{vol}(T^2)$, and τ is such that the metric is

$$ds_2^2 = \frac{\text{Im}\rho}{\text{Im}\tau}|dx^8 + \tau dx^9|^2 \equiv \frac{\text{Im}\rho}{\text{Im}\tau}\, dz d\bar{z} \ .$$

Take $\text{Im}\rho = 1$ and show that one can write $g_2 = \partial_\tau\partial_{\bar{\tau}}K$ where $K = -\log\left(i\int\Omega\wedge\bar{\Omega}\right)$ and $\Omega = dx^8 + \tau dx^9$.

Exercise 2.5:

The group $O(k,k,\mathbb{Z})$ is generated by the following group elements

$$O_\Theta = \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix} \ \Theta_{mn} \in \mathbb{Z}\,, \quad O_M = \begin{pmatrix} M & 0 \\ 0 & (M^t)^{-1} \end{pmatrix} \ M \in GL(k;\mathbb{Z})\,, \quad O_{D_i} = \begin{pmatrix} 1 - D_i & D_i \\ D_i & 1 - D_i \end{pmatrix}\,,$$

where $\Theta^T = -\Theta$ and D_i is a $k \times k$ matrix with all zeros except for a one at the ii component. See how these elements act on the on the momentum and winding numbers (ω^i, n_i), and on the generalized metric $\mathcal{H}$

$$\mathcal{H} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \tag{1}$$

Disentangle these as actions on g and B. How does $\prod_{i=1}^{k} O_{D_i}$ act?

Exercise 2.6:

Take a 3-torus T^3 parameterised by coordinates $(x^1, x^2, x^3) \sim (x^1+1, x^2, x^3) \sim (x^1, x^2+1, x^3) \sim (x^1, x^2, x^3+1)$ with N units of H_3 flux on it. Take a gauge for the B-field such that ∂_{x^3} is an isometry. T-dualize this background along x^3, i.e. transform by the $O(3,3,\mathbb{Z})$ element O_{D_3} in the previous exercise. What are the T-dual metric and B-field? What is the new manifold topologically? Taking the original T^3 as an $S^1 \times T^2$, with the T^2 in the (x^1, x^2) directions, and parameterise the T^2 in terms of the Kähler and complex structure moduli ρ and τ in exercise 2.4. How does the T-duality act on ρ and τ? Do a second T-duality in the direction x^2. Can you make sense of the dual background globally? How does the second T-duality act on τ and ρ?

Exercise 3.1:

Show that $\mathbb{C}P^n$ is Kahler (show it is complex by building an atlas with holomophic transition functions). The Kahler potential is the "Fubini-Study":

$$K = \log(1 + |z|^2)$$

Exercise 3.2:

Consider the group manifold defined by six globally defined 1-forms e^a, $a = 1, ..., 6$ satisfying

$$de^1 = de^2 = de^3 = de^4 = 0 \ , \ de^5 = e^1 \wedge e^2 \ , \ de^6 = e^3 \wedge e^4 \ .$$

Is the manifold complex? Is it symplectic? Is it Kähler? Answer these questions by trying to build explicitly a 3-form Ω and a 2-form J.

Exercise 4.1:

Let $\{\omega_a\}$ be a basis for H^p. Show that there exists a basis $\{\tilde{\omega}^a\}$ for H^{d-p} with the property that

$$\int_M \omega_a \wedge \tilde{\omega}^b = \delta_a{}^b$$

Exercise 4.2:

Compute the Betti numbers for the manifold in Exercise 3.2.

Exercise 4.3:

Show that $\triangle A_p = 0 \Leftrightarrow dA_p = 0, d^\dagger A_p = 0$

Exercise 4.4:

Show that $h^{p,q} = h^{q,p}$ and $h^{n-p,n-q} = h^{p,q}$.

Exercise 5.1:

Consider a compactificaion on $T^2 \times T^2 \times T^2$ with a single complex structure modulus, namely: $dz^i = dx^i + \tau dy^i$, i=1,2,3, where (x^i, y^i) are the coordinates on each torus. Consider the 3-forms $\alpha_0 = dx^1 \wedge dx^2 \wedge dx^3, \alpha_i = \epsilon_{ijk} dx^j \wedge dx^k \wedge dy^i$ (no sum over i), $\beta^i = \epsilon_{ijk} dy^j \wedge dy^k \wedge dx^i$, $\beta^0 = dy^1 \wedge dy^2 \wedge dy^3$. Write Ω in terms of this basis, and find the Kahler potential. Add fluxes for the field strengths H_3 y F_3 along all 3-cycles. Find the potential. Show that there are Minkowski solutions with the modulus τ fixed.