EXERCISES: INFINITE DIMENSIONAL LIE ALGEBRAS

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Exercise 1. Let $A = k[t_1, \dots, t_n]$. Describe the Lie algebra $\tilde{W}_n = DerA$ of derivations of A and describe the structure of A as a module over $\tilde{W}_n$.

Exercise 2. Let $A = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and W_n the Lie algebra of vector fields on n-dimensional torus ($W_n = DerA$). a) Show that W_1 is a simple Lie algebra (it has no two-sided ideals other than 0 and W_1). Deduce that W has no nontrivial finite dimensional modules. b) Consider the 2-parameter families of modules of *intermediate series* over W_1:

$$V(\alpha, \beta) = \sum_{s \in \beta + \mathbb{Z}} \mathbb{C} v_s,$$

where $\alpha, \beta \in \mathbb{C}$ and $(t^{n+1}\frac{d}{dt})v_s = (s + n\alpha)v_{s+n}$. Show that $V(\alpha, \beta)$ is simple unless $\alpha = 0, 1$ and $\beta \in \mathbb{Z}$. Describe the structure of $V(\alpha, \beta)$ in all cases.
c) Establish when $V(\alpha, \beta)$ and $V(\alpha', \beta')$ are isomorphic.

Exercise 3. a) Let V be a simple finite dimensional $\mathfrak{gl}(n)$-module, $\gamma \in \mathbb{C}^n$ and $A = \mathbf{q}^\gamma \mathbb{C}[q_1^{\pm 1}, \dots, q_n^{\pm 1}]$. Verify that the *tensor density module*

$$T(\gamma, V) = A \otimes V$$

has a module structure over the Lie algebra W_n of vector fields on n-dimensional torus (defined in the lecture).
b) Identify $T(\mathbf{0}, \Lambda^k \mathbb{C}^n)$ with the k-forms Ω^k. Show that closed 1-forms form an irreducible submodule of Ω^1.
c) Show that Ω^0 and Ω^n are reducible W_n-modules.
d) Describe the structure of $T(\gamma, V)$ as an A-module.
e) Prove an embedding of $\mathfrak{sl}(n+1)$ into W_n by:

$$E_{ij} \mapsto t_i t_j^{-1} \partial_j, 1 \leq i, j \leq n, \; E_{i,n+1} \mapsto -t_i \sum_{j=1}^{n} \partial_j,$$

$$E_{n+1,i} \mapsto t_i^{-1} \partial_i, \; E_{n+1,n+1} \mapsto -\sum_{j=1}^{n} \partial_j,$$

where $\partial_i = t_i \frac{\partial}{\partial t_i}$.
f) Describe an embedding of $\widehat{\mathfrak{sl}(n+1)}$ into W_{n+1}.

Exercise 4. Let k be an algebraically closed field of characteristic 0, $X \subset \mathbb{A}_k^n$ an irreducible affine variety defined by the ideal $I \subset k[x_1, \dots, x_n]$ and $A = k[x_1, \dots, x_n]/I$ the algebra of polynomial functions on X. Let $\mathcal{D}_X$ be the Lie algebra of (polynomial) vector fields on X. Show that there exists a natural isomorphism between $\mathcal{D}_X$ and

$$\left\{d \in \tilde{W}_n \,\middle|\, d(I) \subset I\right\} / \left\{d \in \tilde{W}_n \,\middle|\, d(k[x_1, \dots, x_n]) \subset I\right\}$$

Exercise 5. Let $I = (f_1, \dots, f_m)$. Identify $\mathcal{D}_X$ with an A-module of solutions in A^n of the system of linear homogeneous equations with the Jacobian matrix coefficients:

$$\mathrm{Jac} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ & \dots & \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Exercise 6. Let $X = S^2$, $A = k[x, y, z] / \left\langle x^2 + y^2 + z^2 - 1 \right\rangle$. Show that the Lie algebra $\mathcal{D}_X$ of polynomial vector fields on X is generated (as an A-module) by the vector fields $\Delta_{ab} = x_b \frac{\partial}{\partial x_a} - x_a \frac{\partial}{\partial x_b}$, $1 \leq a \neq b \leq 3$ subject to the relation

$$x_c \Delta_{ab} + x_a \Delta_{bc} + x_b \Delta_{ca} = 0.$$

Exercise 7. Show that the following map defines an embedding of sl_n into the Lie algebra $\mathcal{D}_X$ of polynomial vector fields on $\mathbb{S}^{n-1}$:

$$E_{ab} \mapsto \sum_{p=1}^{N} x_b x_p \Delta_{ap},$$

$$E_{aa} - E_{bb} \mapsto \sum_{p=1}^{N} x_a x_p \Delta_{ap} - x_b x_p \Delta_{bp}.$$

Exercise 8. Let $\mathcal{H} = \sum_{i \in \mathbb{Z}} \mathbb{C} a_i \oplus \mathbb{C} c$ is a Heisenberg subalgebra.
a) Let $\mathcal{F} = \mathbb{C}[x_1, x_2, \dots,]$ the space of polynomials in infinitely many variables. Show that the formulas:

$$a_i \mapsto r_i \frac{\partial}{\partial x_i},\ a_{-i} \mapsto r_i^{-1} i x_i,\ i > 0$$

$$a_0 \mapsto \lambda I,\ c \mapsto 1$$

define a *Fock space representation* of $\mathcal{H}$ on $\mathcal{F}$ for any choice of scalars λ, r_i. Prove that $\mathcal{F}$ is irreducible.
b) Show that $U(\mathcal{H})/(c - 1)$ is isomorphic to the Weyl algebra with infinitely many generators. Using this fact construct examples of non-highest (non-lowest) weight irreducible $\mathcal{H}$-modules.

Exercise 9. Let L be the Virasoro algebra $L = \sum_{i \in \mathbb{Z}} L_i \oplus \mathbb{C} c$, $M(z, \lambda)$ is the Verma module generated by the vacuum vector v with $L_i v = 0$ for $i > 0$, $L_0 v = \lambda v$ and $cv = zv$, where $z, \lambda \in \mathbb{C}$. Let $L(z, \lambda)$ be the unique irreducible quotient of $M(z, \lambda)$.
a) Compute the dimensions of the weight subspaces $M(z, \lambda)_\mu$ (the character of $M(z, \lambda)$).
b) Consider the restricted dual $L^{\checkmark}(z, \lambda) = \sum_i L(z, \lambda)_i^*$ of $L(z, \lambda)$, where

$xf(v) = -f(xv)$ for $f \in L^{\checkmark}(z,\lambda)$, $v \in L(z,\lambda)$, $x \in L$. Show that $L^{\checkmark}(z,\lambda)$ is an irreducible lowest weight module.
c) Find conditions on λ under which $M(0,\lambda)_i = L(0,\lambda)_i$ for $i = 1,2$.
d) Compute the restrictions of the Shapovalov form on $M(0,\lambda)_i$ for $i = 1,2$ and deduce c).
e) Let $L_k \mapsto \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+k} + i\gamma k a_k$, $k \neq 0$, $L_0 \mapsto \sum_{i>0} a_{-i}a_i + \frac{\lambda^2+\gamma^2}{2}$. Show that this defines a Fock space representation of the Virasoro algebra on $\mathcal{F}$ with central charge $1 + 12\gamma^2$.

Exercise 10. Let $\widehat{\mathfrak{g}}$ be an Affine Kac-Moody algebra.
a) Show that any two irreducible integrable highest weight $\widehat{\mathfrak{g}}$-modules do not have a non trivial extension.
b) Construct examples of irreducible integrable non-highest weight $\widehat{\mathfrak{sl}(2)}$-modules.
c) Let $\widehat{\mathfrak{g}}$ be an Affine Kac-Moody algebra with a degree derivation d. Show that $\widehat{\mathfrak{g}}$ does not have finite dimensional representations.

Exercise 11. For a finite dimensional module V over $\mathfrak{g}$ and a nonzero complex number a define an *evaluation* module $V(a)$ over the *loop algebra* $\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$ as follows: $V(a) = V$ as a vector space and $(x \otimes t^n)v = a^n xv$, for any $x \in \mathfrak{g}$, $v \in V$.
a) When $V(a)$ is isomorphic to $V(b)$?
b) Let $V_1,\ldots,V_m$ irreducible nonzero $\mathfrak{g}$-modules, $a_1,\ldots,a_m \in \mathbb{C}$. Denote

$$V(a_1,\ldots,a_m) = V_1(a_1) \otimes \ldots \otimes V_m(a_m).$$

Under what condition on the parameters $a_1,\ldots,a_m$ the module $V(a_1,\ldots,a_m)$ is irreducible?