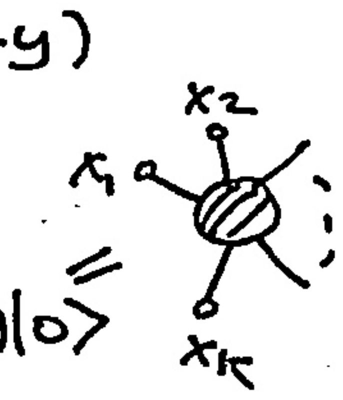
Lectures in B.A. – october 2019: ~~[illegible]~~ Renormalization QFT and ~~general~~ cogroup/coloop bialgebras

Lecture 1 – ~~[illegible]~~ Renormalization ~~[illegible]~~ in QFT

1. Physical context: QFT (brief overview)

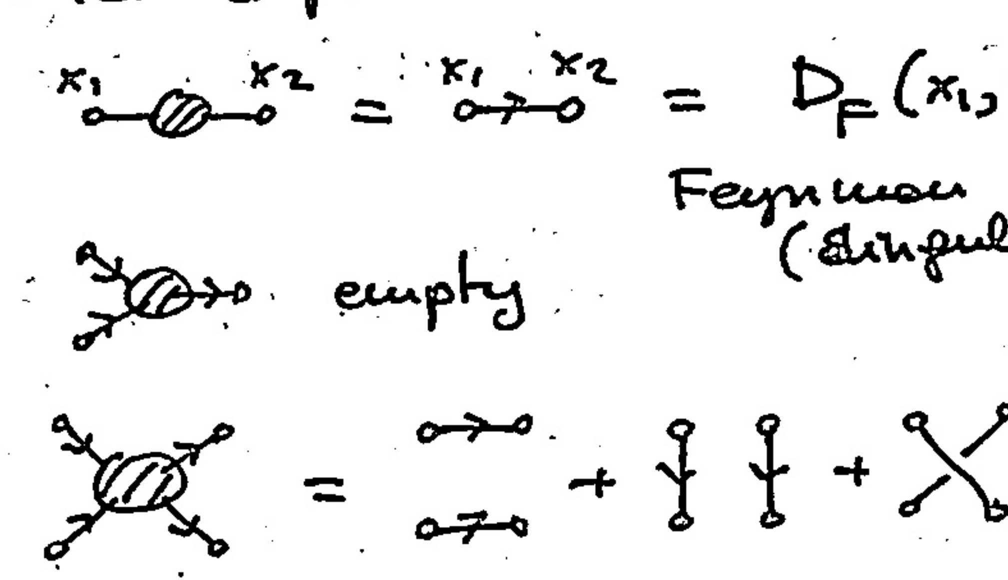
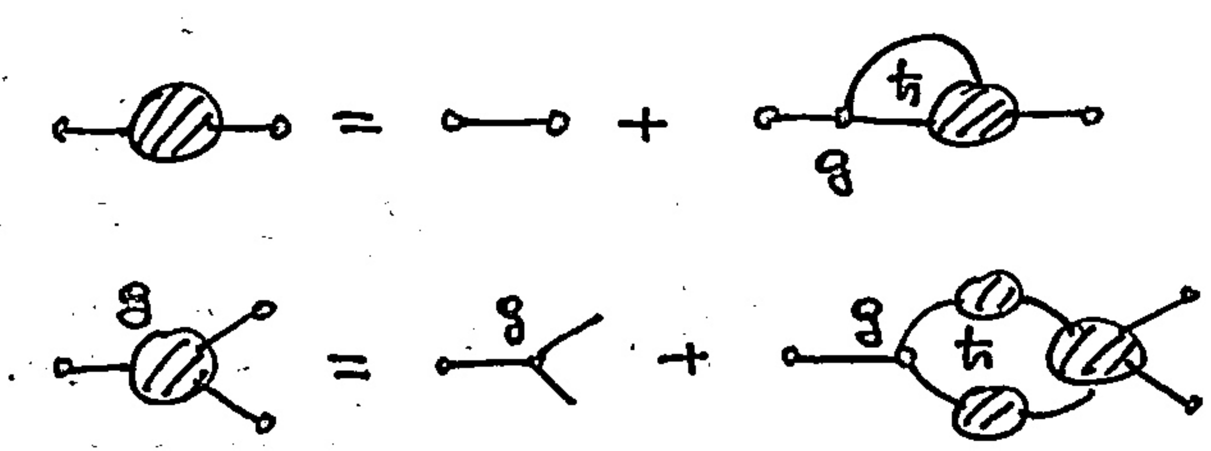
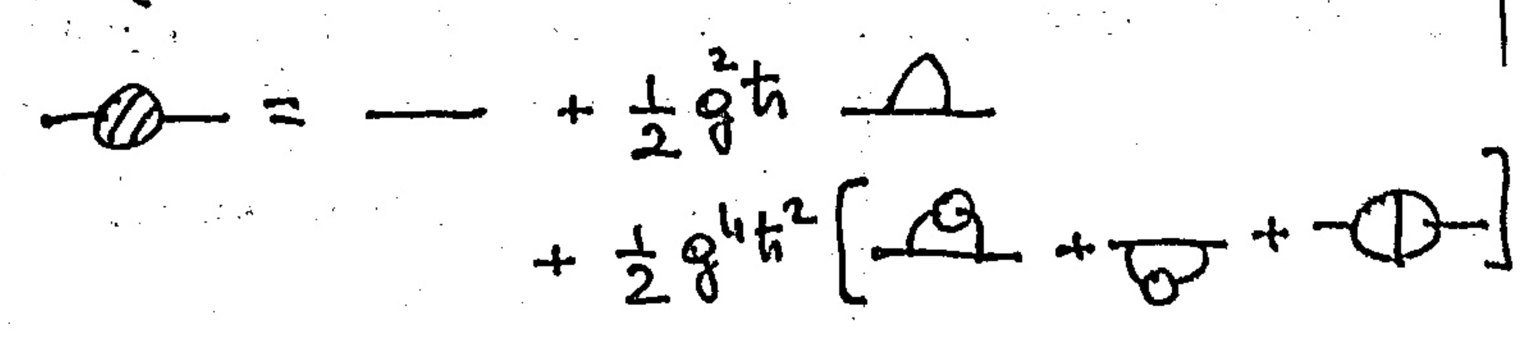
Classical $\xrightarrow{\hbar}$ **Quantum**

Classical		Quantum
• matter fields $\psi, \bar\psi : M \to E$ spin or $\phi = \bar\phi$; gauge fields $A^a_\mu \in \Omega^1_{\mathfrak{L}}(M, \mathfrak{g})$ • Lagrangian $\mathcal{L}(\phi, \partial\phi)$ $\Rightarrow$ Euler-Lagrange eqt. • observables = functions gen. by ϕ and $\pi = \dot\phi$	$\xrightarrow{\hbar}$	• field opts $\hat\phi(x), \hat\psi(x), \hat A^a_\mu(x)$ on Hilbert space s.t. canonical comm. rel (Heisenberg uncertainty principle) $[\hat\phi(x), \hat\pi(y)] = i\hbar\,\delta(x-y)$ • correlation fcts $\langle \hat\phi(x_1)\cdots\hat\phi(x_k)\rangle := \langle 0 \vert T\hat\phi(x_1)\cdots\hat\phi(x_k)\vert 0\rangle$ =  $\times$ use Feynman graphs ~~[illegible]~~ rules for amplitude.

free theory

g coupling constant (charge) $\downarrow$

(self-) interacting theory

$\mathcal{L} = \mathcal{L}_{kin}$ e.g. Klein-Gordon $-\frac{1}{2}(\partial_\mu\bar\psi\partial^\mu\psi + m^2\bar\psi\psi)$ Dirac $\bar\psi(i\gamma^\mu\partial_\mu - m)\psi$ Maxwell $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$	$\xrightarrow{\hbar}$	Wick expansion:  $= D_F(x_1, x_2)$ Feynman propagator (singular distribution); empty; etc.
$\mathcal{L} = \mathcal{L}_{kin} + g\,\mathcal{L}_{int}$ eg. scalar: $\frac{g}{3!}\phi^3$ or $g\,\phi^4$ QED: $\alpha\,\bar\psi\gamma^\mu A_\mu\psi$ (type ϕ^3) $\alpha = e^2/4\pi\hbar c \sim \frac{1}{137}$ QCD: $G^a_{\mu\nu} = F^a_{\mu\nu} + g\,f^{abc}A^b_\mu A^c_\nu$ (type ϕ^4)	$\overset{\hbar?}{\dashrightarrow}$	Schwinger-Dyson eqts (e.g. for proper 1PI fcts)  etc (similar for connected fcts) • exact solution not known • perturbative sol. for small g possible ("asymptotic freedom" at small length scale)  $\Rightarrow$ formal series in powers of g: $G_k(x_1, \dots, x_k; g, m) = \sum_{n=0}^{\infty} G_n(x_1, \dots, x_k; m)\, g^{2n}$ (*) $= \sum_\Gamma \frac{\hbar^{L(\Gamma)}}{\mathrm{sym}(\Gamma)} U(x_1..x_k; \Gamma)\, g^{V(\Gamma)}$

Ref: – brief intro: M. Dasgupta (2008)
J. Polonyi (2014)

– standard books: Peskin and Schröder
Itzykson and Zuber
Bjorken and Drell ...
Cottingham and Greenwood (Standard Model)

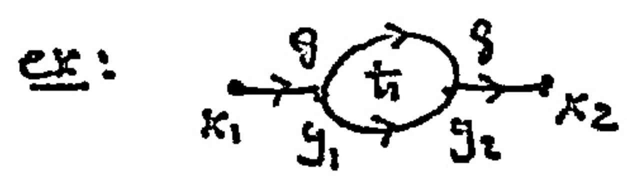
(*) Evidence: perturbative series explain "(vertex) corrections" in measured charge.
E.g. electrostatical pot. btw charges q, q' at distance r: $V(r) = \frac{qq'}{4\pi}\cdot\left[1 + e^2\cdot f(r) + \cdots\right] \neq$ Coulomb potential!
(bare charges which polarize a cloud) $\frac{q_0 q_0'}{4\pi} \xleftarrow{r\to 0}$; $f(r) \xrightarrow{r\ \mathrm{large}} 0$

2)

[Dyson '49, Collins, Itzykson & Zuber]

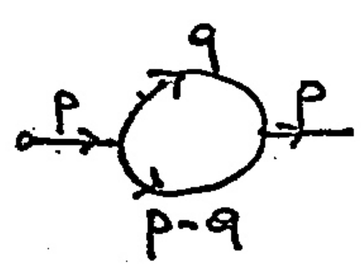
2. Renormalization : Dyson's formulas and diffeomorphism group

- Pb with "infinites" or with product of singular distributions (cf Estanislao's lectures)

ex:  $= \frac{\hbar g^2}{2} \int_{M^2} D_F(x_1,y_1)\, D_F(y_1,y_2)^2\, D_F(y_2,x_2)\, dy_1\, dy_2$

square not defined on singularity $\{y_1 = y_2\}$

If M is flat, Fourier transform to momentum space : $D_F(y_1,y_2) \mapsto \frac{1}{q^2+m^2}$

 $= \frac{\hbar g^2}{2} \int_{q^2=m^2}^{\infty} \frac{d^Dq}{(2\pi)^D} \frac{1}{q^2+m^2} \frac{1}{(p-q)^2+m^2} \overset{\text{spherical}}{\sim} \int \frac{d^D|q|}{|q|^4} \sim \int \frac{d|q|}{|q|^{4-(D-1)}}$

- in D=4 : $\sim \int_{|q|_{min}}^{\infty} \frac{d|q|}{|q|} = \left[\log|q|\right]_{|q|_{min}}^{\infty}$ ← logarithmic divergency

- in D=6 : $\sim \int_{|q|_{min}}^{\infty} |q|\, d|q| = \left[|q|^2\right]_{|q|_{min}}^{\infty}$ ← quadratic divergency

1st ⟹ need to subtract divergencies to compute the vertex corrections of g, e, m.

- So, the physical parameters g, e, m in $\mathcal{L}$ are not the measured ones! ~~Call them "bare"~~! Call them "bare" : g_0, e_0, m_0 - Need expression in terms of effective g, e, m!

bare Green fcts $\quad G(p_1,..,p_k;\, g_0, m_0) = \sum_{\Gamma} \frac{\hbar^{L(\Gamma)}}{\mathrm{sym}(\Gamma)} U_p(\Gamma)\, g_0^{V(\Gamma)} = \sum_n G_n(p)\, g_0^{2n}$

renormalized Green fcts $\quad G^{ren}(p;\, g, m) = \sum_{\Gamma} \frac{\hbar^{L(\Gamma)}}{\mathrm{sym}(\Gamma)} R_p(\Gamma)\, g^{V(\Gamma)} = \sum_n G_n^{ren}(p)\, g^{2n}$

Dyson '49 (for QED, with Ward id.)

Theory is renormalizable if $\mathcal{L}^{ren}(\phi; g, m) := \mathcal{L}(\phi; g, m) + \Delta\mathcal{L} \overset{!}{=} \mathcal{L}(\phi_0;\, g_0, m_0)$ "self-similarity" at different length/energy scales

where set
$$\boxed{\begin{aligned} \phi_0 &= Z_3^{1/2}(g)\,\phi \\ m_0^2 &= m^2 + \delta m^2(g) = m^2 \cdot Z_m(g^2)\, Z_3(g^2)^{-1} \\ g_0 &= g \cdot Z_1(g^2) \cdot Z_3(g^2)^{-3/2} \quad (\text{for } \phi^3 \text{ type}) \end{aligned}}$$

[cf. Gell-Mann and Low '54]

Then : $\boxed{G^{ren}(x_1,..,x_k;\, g, m) = G(x_1,..,x_k;\, g_0(g), m(g)) \cdot Z_3(g^2)^{-k/2}}$

where the renormalization factors are invertible series $Z(g^2) = 1 + O(g^2) + O(g^4) \ldots$
and the coupling constant^2 is a formal diffeomorphism $g_0(g)^2 = g^2 + O(g^4) + \ldots$

3. Perturbative renormalization group

- algebraic structure of (perturbative) renormalization group ~~semidirect product~~

$$\boxed{\underset{(\mathrm{Dif},\,\circ)}{\text{Coupling constant}} \ltimes \underset{(\mathrm{Inv},\,\cdot)}{\text{Ren. factors}}}$$

acting on Green's fcts. and on Lagrangians.

with semidirect group law : $(g_0, Z) \circ (g_1, Z') = (\underbrace{g_0 \circ g_1}_{g_2},\, \underbrace{(Z \circ g_1) \cdot Z'}_{Z''})$

- groups :

$$\boxed{\begin{aligned} \mathrm{Dif} &= \Big\{ a(g) = g + \sum_{n=1}^{\infty} a_n\, g^{n+1},\ a_n \in \mathbb{C} \Big\}, \circ \\ \mathrm{Inv} &= \Big\{ b(g) = 1 + \sum_{n=1}^{\infty} b_n\, g^n,\ b_n \in A \Big\}, \cdot \end{aligned}}$$

effective coupling ⟺ Lagrange inversion in a Schröder eqt [Schröder]

algebra $A = \mathbb{C}$ for ϕ^3, ϕ^4

$A = M_4(\mathbb{C})$ for ψ, A_μ

- Here : effective coupling by Lagrange inversion in an implicit eqt [Schröder 1871, Koenigs 1884, Poincaré 1890, ...]

Lecture 2 – Renormalization Hopf algebras and functoriality

4. Faà di Bruno Hopf algebra of formal diffeomorphisms [Ref: Rota & students, BFK 2006]

- ~~[illegible]~~ Groups of series are proalgebraic i.e. representable functors $\mathrm{Diff}, \mathrm{Inv}: \underline{\mathrm{Com}}_k \to \underline{\mathrm{Group}}$ repr. by commutative, unital, associative Hopf algebras: $\forall A \in \mathrm{Com}_k$

1) $\boxed{\begin{array}{c}\mathrm{Diff}(A) \cong \mathrm{Hom}_{\mathrm{Com}_k}(\mathcal{H}^{FdB}, A) \\ \circ \longleftrightarrow \text{convolution product } *_{\Delta^{FdB}}\end{array}}$ where $\boxed{\begin{array}{l}\mathcal{H}^{FdB} = k[x_n, n \geq 1] \quad \text{Faà di Bruno Hopf alg} \\ \Delta^{FdB}(x_n) = \sum_{m=0}^{n} x_m \otimes \sum_{\substack{k_0+\dots+k_m=n-m \\ k_i \geq 0}} x_{k_0} x_{k_1} \cdots x_{k_m} \\ \varepsilon(x_n) = \delta_{n,0} \quad (x_0 \equiv 1).\end{array}}$

Key point:
Lagrange inversion in $\mathrm{Diff}(A) \Longleftrightarrow$ antipode in $\mathcal{H}^{FdB}$ ~~[illegible]~~
↳ allows to find Lambert's W fct def by $W(u) = t$ s.t. $u = te^t$
as a series $W(u) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^n}{(n+1)!} u^{n+1}$ with convergence radius $1/e$, used to compute g_0, cf [Curtright, Zachos 2010]

2) $\boxed{\begin{array}{c}\mathrm{Inv}(A) \cong \mathrm{Hom}_{\mathrm{Com}_k}(\mathcal{H}^{Inv}, A) \\ \circ \longleftrightarrow \text{convol. pr. } *_{\Delta^{Inv}}\end{array}}$ where $\boxed{\begin{array}{l}\mathcal{H}^{inv} = k[x_n, n \geq 1] \quad \text{Hopf algebra of symmetric fcts} \\ \Delta^{inv}(x_n) = \sum_{m=0}^{n} x_m \otimes x_{n-m} \quad (x_0 = 1)\end{array}}$

- Then: couplings, ~~and~~ ren. factors ~~[illegible]~~ and even Green's fcts are characters (alg. algebra morphism) of $\mathcal{H}^{FdB}$ or $\mathcal{H}^{inv}$ with values in suitable $A \in \mathrm{Com}_k$ e.g. $A = \mathbb{C}$ over $k = \mathbb{C}$ or $A = \mathbb{C}[[\varepsilon, \varepsilon^{-1}]]$ (Laurent pol.) for regularized amplitudes.

5. BPHZ ~~[illegible]~~ renormalization formula [Ref: Zimmermann '69, Collins] s.t. $\mathcal{L}^{ren}(\phi, g, m) = \mathcal{L}(\phi_0, g_0, m_0)$

- Pb in Dyson's formula: given (g_0, m_0), how to find Z's to describe (g, m) measured ~~[illegible]~~ ?
- Need counterterms of divergent Feynman graphs: $\boxed{\begin{array}{l} Z_m(g) = 1 + \sum_{res(\Gamma) = -} C_0(\Gamma)\, g^{V(\Gamma)} \\ Z_3(g) = 1 + \sum_{res(\Gamma) = -} C_2(\Gamma)\, g^{V(\Gamma)} \\ Z_1(g) = 1 + \sum_{res(\Gamma) = \prec} C_0(\Gamma)\, g^{V(\Gamma)-1}\end{array}}$

- BPHZ ('57, '66, '69): – subtract Taylor expansion to divergency order
  – recursive formula on subdivergencies

$$\boxed{\begin{array}{l}\forall r = 0, \dots, \mathrm{div}(\Gamma): \quad C_r(\Gamma) = -\frac{1}{r!} \partial_p^r R_p^{prep}(\Gamma)\big|_{p=0} \\ R_p^{prep}(\Gamma) = U_p(\Gamma) + \sum_{\substack{\gamma_1, \dots, \gamma_\ell \subsetneq \Gamma \\ 1PI, \mathrm{div}, \gamma_i \cap \gamma_j = \emptyset}} \sum_{r_1, \dots, r_\ell} U_p\left(\Gamma / \{\gamma_i(r_i)\}\right) C_{r_1}(\gamma_1) \cdots C_{r_\ell}(\gamma_\ell) \\ \qquad \text{contracted graph } \gamma_i \text{ replaced by } \overset{(r_i)}{-\!\!\times\!\!-} \\ R_p(\Gamma) = R_p^{prep}(\Gamma) + \sum_{r=0}^{\mathrm{div}(\Gamma)} p^r C_r(\Gamma)\end{array}}$$

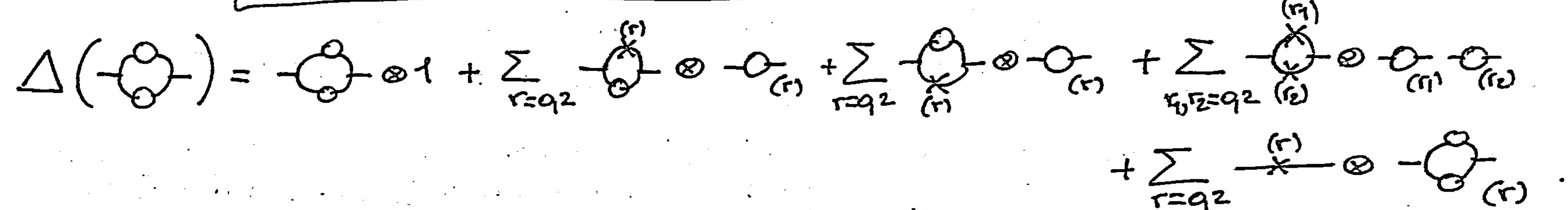
Ex ($D=6$): $R_p(-O-) = U_p(-O-) - U_p(-O-)|_{p=0} - \frac{1}{2} p^2 \frac{\partial^2}{\partial p^2} U_p(-O-)|_{p=0}$
$= U_p(-O-) + m^2 C_0(-O-) + p^2 C_2(-O-)$
$= U_p(-O-)\, C_0(1) + U_p(\overset{(0)}{-\!\!\times\!\!-})\, C_0(-O-) + U_p(\overset{(2)}{-\!\!\times\!\!-})\, C_2(-O-)$.

4)

6. Connes-Kreimer Hopf algebra of Feynman graphs

- Kreimer '98, Connes-Kreimer '00 :

$$\boxed{\begin{array}{l}\text{There exists a Hopf algebra } \mathcal{H}^{CK} = \mathbb{C}[\text{1PI, div } \Gamma] \quad (\text{for } \phi^3) \\ \text{with } \Delta^{CK}(\Gamma) = \Gamma\otimes 1 + \sum_{\substack{\{\gamma_i\} \\ \{r_i\}}} \Gamma/_{\{\gamma_i(r_i)\}} \otimes \gamma_{1(r_1)}\cdots\gamma_{\ell(r_\ell)} + \sum_{r=0}^{\text{div}(\Gamma)} \overset{(r)}{-\!\!\times\!\!-} \otimes \Gamma_{(r)} \\ \text{s.t.: 1) } R_p(\Gamma) = \langle U_p \otimes C \mid \Delta^{CK}(\Gamma)\rangle = (U_p * C)(\Gamma) \\ \qquad \text{i.e. renormalization given by convolution product of } \Delta^{CK} \\ \qquad \text{2) } C(\gamma_{(r)}) := C_r(\gamma) \text{ is related to the antipode} \\ \qquad\quad \text{and define a character of } \mathcal{H}^{CK} \text{ (algebra morphism)}\end{array}}$$

ex: 

- Therefore : 1) Charge renormalization is computed in the group of diffeographisms $\boxed{\text{Difg} := \text{Hom}_{\text{Comm}}(\mathcal{H}^{CK}, \mathbb{C}[[\varepsilon,\varepsilon^{-1}]])}$ ← values in reg. algebra

2) Contains "unphysical" series $\boxed{a(g) = \sum_{\substack{\text{1PI, div }\Gamma \\ \text{res}(\Gamma)=\prec}} a(\Gamma)\, \text{"}g^\Gamma\text{"}}$ symbol

composed with convolution product $*_{\Delta^{CK}}$.

3) Feynman graphs are natural virtual local coordinates in QFT!

Feynman rules: ⟺ connected graphs = junction of 1PI graphs

⟹ junction = disjoint union = free product in $\mathcal{H}^{CK}$.

4) Green fcts = characters of the algebra gen. by connected graphs (not only 1PI).

7. Physical and unphysical coupling renormalization

- Connes-Kreimer '00 : $\boxed{\begin{array}{c}\mathcal{H}^{FdB} \xhookrightarrow{\text{Hopf}} \mathcal{H}^{CK} \\ x_n \longmapsto X_n = \sum_{V(\Gamma)=2n+1} \Gamma\end{array}}$ hence $\boxed{\begin{array}{c}\text{Difg} \xrightarrow{\text{group}} \text{Diff} \\ \sum_{\substack{\text{1PI }\Gamma\\ \text{div}}} a(\Gamma)\, g^\Gamma \longmapsto \sum_{n=0}^{\infty} \underbrace{\Big(\sum_{V(\Gamma)=2n+1} a(\Gamma)\Big)}_{a_{2n}} g^{2n+1}\end{array}}$

unphysical, need for vertex correction/counterterms — physical ⟹ beta fct $\beta(g)$, RG eqt

[Gell-Mann and Low '54, Wilson '71, '75, Nobel '82, Callan-Symanzik '70]

8. Functoriality

- Extra property: perturbative ren. groups are proalgebraic!

i.e. 1) functors $G : \text{Com}_k \to \text{Groups}$ representable by Hopf alg : $G(A) = \text{Hom}_{\text{Com}}(\mathcal{H}, A)$

2) series $g_0(g)$, $Z(g)$ are characters of $\mathcal{H}$ with values in suitable alg. $A \in \text{Com}$

3) compute in $\mathcal{H}$ and apply to specific A.

- Pb: a priori functoriality fails if series have coeff in a non-comm. algebra, e.g. for QED and QCD when amplitude of Feynman graphs $\in M_4(\mathbb{C})$!
- Aim: extend proalg gps Diff, Inv, Difg, Invg to cat. of non-commutative algebras.

Lecture 3 – Cogroup and coloop bialgebras

Aim: describe renormalization groups Inv and Diff as functors on associative algebras (non comm).

9. Groups and loops

- Functors of algebra A over a field k: $\boxed{\begin{array}{l} \mathrm{Inv}(A) = \{ a(g) = \sum_{n=0}^{\infty} a_n g^n \mid a_0=1, a_n\in A \} \text{ with product } \circ \\ \mathrm{Diff}(A) = \{ a(g) = \sum_{n=0}^{\infty} a_n g^{n+1} \mid a_0 = 1, a_n \in A \} \text{ with composition } \circ \end{array}}$

- Applied to $A \in \underline{\mathrm{Com}_k}$ (unital commutative associative algebras$/k$) give:
  - $\mathrm{Inv}(A)$ abelian group rep. by $H^{\mathrm{inv}} = k[x_n \mid n\geq 1]$ $(x_0=1)$
    $\Delta^{\mathrm{inv}}(x_n) = \sum_{m=0}^{n} x_m \otimes x_{n-m}$
  - $\mathrm{Diff}(A)$ group rep. by $H^{\mathrm{FdB}} = k[x_n \mid n\geq 1]$ $(x_0=1)$
    $\Delta^{\mathrm{FdB}}(x_n) = \sum_{m=0}^{n} x_m \otimes \sum_{k_0+\dots+k_m = n-m} x_{k_0}\cdots x_{k_m}$

- Applied to $A\in \underline{\mathrm{As}_k}$ (unital associative algs) give:
  - $\mathrm{Inv}(A)$ group (non abelian)
  - $\mathrm{Diff}(A)$ not a group because composition is not associative: loop!
    $\big((a\circ b)\circ c - a\circ(b\circ c)\big)(g) = (a_1b_1c_1 - a_1c_1b_1)g^4 + \cdots \neq 0$

<u>Def</u>: A <u>loop</u> is a set Q with a multiplication $\circ$, a unit 1 and left/right divisions $a\backslash b$, a/b s.t. <u>cancellation property</u>:
$$a\cdot(a\backslash b) = b \quad\text{and}\quad a\backslash(a\cdot b) = b \qquad \forall a,b\in Q.$$
$$(a/b)\circ b = a \quad\text{and}\quad (a\circ b)/b = a$$

A <u>morphism of loops</u> is a map $f: Q\to Q'$ preserving $\circ, 1, \backslash, /$.

Denote the cat. of loops by <u>Loop</u>.

<u>Ex</u>: 1) <u>Moufang loops</u> [R. Moufang '35] with extra properties: $a(b(ac)) = ((ab)a)c$ + three more
2) S^7 = unit norm octonions is a loop (compare with $S^3\cong SU(2)$ group, $S^1 = U(1)$ abelian gp)

<u>Prop</u>: 1) In a loop Q, left/right inverses exist: $1/a$ st. $(1/a)\circ a = 1$ and $a\backslash 1$ st. $a\circ(a\backslash 1) = 1$ but do not necessarily coincide, and do not reconstruct divisions via $a/b = a\circ(1/b)$ and $a\backslash b = (a\backslash 1)\circ b$ $(*)$

2) Associative loops are groups: $1/a = a\backslash 1$ and $(*)$ holds.

10. Cogroups and coloops in a category C

<u>Def</u>: A functor
$$\begin{tikzcd} C \arrow[r] \arrow[dr] & \underline{\mathrm{Loop}} \text{ or } \underline{\mathrm{Group}} \arrow[d] \\ & \underline{\mathrm{Set}_*} \end{tikzcd}$$
is <u>representable</u> if its composite to pointed sets (sending $\{1\}$ to $\{*\}$) is representable.

For this, need:

1) <u>Yoneda functor</u> $\quad \mathrm{RepFunct}(C, \mathrm{Set}) \overset{\sim}{\longleftrightarrow} C$ (equivalence of cat.)
$$Y(H) = \mathrm{Hom}_C(H,-) \longleftarrow H$$

2) Assume that C has <u>initial object</u> I, i.e. $\forall A\in \mathrm{Ob}\, C$ $\exists!$ $I \xrightarrow{u_A} A$ s.t. $\forall f$ diagram commutes
$$\begin{tikzcd} I \arrow[r, "u_A"] \arrow[dr, "u_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

Then the constant functor $\mathbb{1}: C \to \{*\}$ is rep. by I, i.e. $Y(I) = \mathbb{1}$ (gives the unit in $Y(H)$)

<u>Ex</u>: in Set_*: $I = \{*\}$, in $\underline{\mathrm{Com}_k}$, $\underline{\mathrm{As}_k}$, $\underline{\mathrm{Alg}_k}$ (unital algebras): $I = k$.

3) <u>Assume that</u> C has <u>coproduct</u> $\sqcup$, i.e. $\forall A,B\in \mathrm{Ob}\, C$:
$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[dr, "\forall f"'] & A\sqcup B \arrow[d, "\exists! \langle f, g\rangle"] & B \arrow[l, "i_2"'] \arrow[dl, "\forall g"] \\ & C & \end{tikzcd}$$

Then the $Y(H_1)\times Y(H_2)$ is representable $= Y(H_1\sqcup H_2)$
(~~it~~ allows to rep. the ~~a~~ multiplication in $Y(H)$).

<u>Ex</u>: In Set_*: $\sqcup = \vee_*$, in $\underline{\mathrm{Com}_k}$: $\sqcup = \otimes_k$, in $\underline{\mathrm{As}_k}$: $\sqcup = *$ free product (as in Groups), in $\underline{\mathrm{Alg}_k}$: $\sqcup$ = free product with parenthesizing.

Rem : In a cat $(\mathcal{C}, \sqcup, I)$:

1) $\forall f : A \to A'$, $g : B \to B'$, $\exists\, f \sqcup g = \langle i_1' f, i_2' g \rangle : A \sqcup B \longrightarrow A' \sqcup B'$
2) $\exists\, \sigma : A \sqcup B \longrightarrow B \sqcup A$ def $\sigma = \langle i_2, i_1 \rangle$ symmetry opt s.t. $(\mathcal{C}, \sqcup, I, \sigma)$ is a strict symmetric monoidal category.
3) $\forall A$ $\exists$ folding map $\mu_A = \langle id, id \rangle : A \sqcup A \longrightarrow A$ associative, commutative, with unit u_A, s.t. $\langle f, g \rangle = \mu_A (f \sqcup g)$ $\forall f, g : A \to A$. $\Rightarrow$ Every object is an abelian monoid!

Thm [D. Kan '58 for $\mathcal{C}$ = Group, B. Eckmann and P. Hilton '61–'63]

$$\mathrm{RepFunct}(\mathcal{C}, \underline{\mathrm{Groups}}) \simeq \left\{ \begin{array}{l} \underline{\text{cogroups}} \text{ in } \mathcal{C} := \text{objects } H \text{ in } \mathcal{C} \text{ with} \\ \quad \Delta : H \to H \sqcup H \quad \text{satisfying usual Hopf algebra axioms} \\ \quad \varepsilon : H \to I \quad\quad\quad \text{with } \mu : H \sqcup H \to H \text{ and } u : I \to H \\ \quad S : H \to H \end{array} \right\}$$

Thm [A.F. and I. Shestakov '18]

$$\mathrm{RepFunct}(\mathcal{C}, \underline{\mathrm{Loops}}) \simeq \left\{ \begin{array}{l} \underline{\text{coloops}} \text{ in } \mathcal{C} := \text{objects } H \text{ in } \mathcal{C} \text{ with} \\ \quad \Delta : H \to H \sqcup H,\ \varepsilon : H \to I \text{ and} \\ \quad \delta_\ell, \delta_r : H \to H \sqcup H \text{ satisfying cocancellation property} \end{array} \right\}$$

11. Invertible series and formal diffeomorphisms with associative coefficients

Def A loop/group is proalgebraic if it is a representable functor on a cat. $\mathcal{C}$ = variety of algebras$/k$.

Notation : in $H \sqcup H$, for a generator x of H set : $x \equiv x^{(1)} := x$ in 1st copy of H ($\Leftrightarrow x \otimes 1$ in $H \otimes H$)
$y \equiv x^{(2)} := x$ in 2nd copy of H ($\Leftrightarrow 1 \otimes x$ in $H \otimes H$)

Prop (easy): The proalgebraic group Inv extend to $\underline{As_k}$ and is rep. by the cogroup $H^{inv} = k\langle x_n | n \geq 1 \rangle$ (non com pol.) with $\Delta(x_n) = \sum_{m=0}^{n} x_m y_{n-m} \in H^{inv} \sqcup H^{inv}$.

Thm The proalgebraic group Diff extend to $\underline{As_k}$ as a proalgebraic loop, with coloop $H^{FdB} = k\langle x_n | n \geq 1 \rangle$

$$\Delta(x_n) = x_n + y_n + \sum_{\ell=1}^{n-1} \sum_{\underline{n} \in \mathcal{C}_n^{\ell+1}} \binom{n_1+1}{\ell} x_{n_1} y_{n_2} \cdots y_{n_{\ell+1}} \quad (\text{as in } \underline{Com_k})$$

$$\delta_r(x_n) = x_n - y_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\underline{n} \in \mathcal{C}_n^{\ell+1}} d_\ell(n_1, \ldots, n_\ell) (x_{n_1} - y_{n_1}) y_{n_2} \cdots y_{n_{\ell+1}} \quad (\text{as in } \underline{Com_k})$$

$$\delta_\ell(x_n) = y_n - x_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\underline{n} \in \mathcal{C}_n^{\ell+1}} \sum_{\underline{e} \in \mathcal{E}^\ell} (-1)^{\underline{e}} d_\ell^{\underline{e}}(n_1, \ldots, n_\ell)\, x_{n_1}^{(e_1)} x_{n_2}^{(e_2)} \cdots x_{n_\ell}^{(e_\ell)} (y_{n_{\ell+1}} - x_{n_{\ell+1}})$$

(deeply different than $\underline{Com_k}$!)

where • $\mathcal{C}_n^{\ell+1} = \{ \underline{n} = (n_1, \ldots, n_{\ell+1}) \mid \sum n_i = n,\ n_i \geq 1 \}$ compositions of n of length $\ell+1$

$\underline{ex}$: $\mathcal{C}_3^1 = \{(3)\}$, $\mathcal{C}_3^2 = \{(2,1), (1,2)\}$, $\mathcal{C}_3^3 = \{(1,1,1)\}$

• $\boxed{d_\ell(n_1, \ldots, n_\ell) = \sum_{\underline{m} \in \mathcal{M}_\ell} \binom{n_1+1}{m_1} \binom{n_2+1}{m_2} \cdots \binom{n_\ell+1}{m_\ell}}$ coefficients of Lagrange inversion formula

$\mathcal{M}_\ell = \{ \underline{m} = (m_1, \ldots, m_\ell) \mid m_1 + \cdots + m_\ell = \ell,\ m_1 + \cdots + m_j \geq j\ \forall j = 1, \ldots, \ell-1 \} \cong PBT_{\ell+1}$ planar binary trees

$\underline{ex}$: $\mathcal{M}_2 = \{(2,0), (1,1)\}$, $\mathcal{M}_3 = \{(3,0,0), (2,1,0), (2,0,1), (1,2,0), (1,1,1)\}$

$d_2(n_1, n_2) = \binom{n_1+1}{2} + \binom{n_1+1}{1}\binom{n_2+1}{1}$

$d_3(n_1, n_2, n_3) = \binom{n_1+1}{3} + \binom{n_1+1}{2}\binom{n_2+1}{1} + \binom{n_1+1}{2}\binom{n_3+1}{1} + \binom{n_1+1}{1}\binom{n_2+1}{2} + \binom{n_1+1}{1}\binom{n_2+1}{1}\binom{n_3+1}{1}$.

Furthermore: ◦ $\mathcal{E}_\ell = \{ \underline{e} = (e_1, e_2, .., e_\ell) \mid e_i \in \{1,2\} \}$

$\mathcal{M}_\ell^{\underline{e}} = \{ \underline{m} \in \mathcal{M}_\ell \mid m_i = 0 \text{ if } e_i = 2 \text{ , for } i = 2, .., \ell \}$

ex: $\mathcal{M}_2^{(1,1)} = \mathcal{M}_2 = \{ (2,0), (1,1) \}$ but $\mathcal{M}_2^{(1,2)} = \{ (2,0) \}$

$\mathcal{M}_3^{(1,1,1)} = \mathcal{M}_3 = \{ (3,0,0), (2,1,0), (2,0,1), (1,2,0), (1,1,1) \}$

but: $\mathcal{M}_3^{(1,1,2)} = \{ (3,0,0), (2,1,0), (1,2,0) \}$, $\mathcal{M}_3^{(1,2,1)} = \{ (3,0,0), (2,0,1) \}$, $\mathcal{M}_3^{(1,2,2)} = \{ (3,0,0) \}$

and set $\boxed{ d_\ell^{\underline{e}}(n_1, .., n_\ell) = \sum_{\underline{m} \in \mathcal{M}_\ell^{\underline{e}}} \binom{n_1+1}{m_1} \cdots \binom{n_\ell+1}{m_\ell} }$ new Lagrange-type coefficients

ex: $d_2^{(1,2)}(n_1, n_2) = \binom{n_1+1}{2}$; $d_2^{(1,1)}(n_1, n_2) = d_2(n_1, n_2) = \binom{n_1+1}{2} + \binom{n_1+1}{1}\binom{n_2+1}{1}$.

◦ finally: $(-1)^{\underline{e}} = (-1)^{e_1 + \dots + e_\ell - \ell}$ and $x_{n_i}^{(e_i)} = \begin{cases} x_n & \text{if } e_i = 1 \\ y_n & \text{if } e_i = 2 \end{cases}$.

Proof of thm: stronger assertion showing a rich algebraic and combinatorical structure:

$$\Delta(x_n) = x_n + y_n + \sum_{\ell=1}^{n-1} \sum_{\underline{n} \in \mathcal{E}_n^{\ell+1}} x_{n_1} \triangleright (y_{n_2} \otimes \cdots \otimes y_{n_{\ell+1}})$$

$$\delta_r(x_n) = x_n - y_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\underline{n} \in \mathcal{E}_n^{\ell+1}} (x_{n_1} - y_{n_1}) \triangleright R_\ell(y_{n_2}, .., y_{n_{\ell+1}})$$

$$\delta_\ell(x_n) = y_n - x_n + \sum_{\ell=1}^{n-1} (-1)^\ell \sum_{\underline{n} \in \mathcal{E}_n^{\ell+1}} \sum_{\underline{e} \in \mathcal{E}_\ell} (-1)^{\underline{e}} x_{n_1}^{(e_1)} \triangleright R_\ell^{\underline{e}}(x_{n_2}^{(e_2)}, .., x_{n_\ell}^{(e_\ell)}, y_{n_{\ell+1}} - x_{n_{\ell+1}})$$

where $\triangleright, R_\ell, R_\ell^{\underline{e}}$ are operators defined on any positively graded algebra $A \left(= \overline{T}(X^{(1)} \oplus X^{(2)})\right)$

$\triangleright : T(A) \otimes T(A) \to T(A)$ from $a \triangleright (b_1 \otimes \cdots \otimes b_\ell) = \binom{|a|+1}{\ell} a b_1 \cdots b_\ell$

s.t. $\triangleright$ restricted to $A \otimes T(A) \to A$ is a brace product (cf. operad pre-Lie)

and $R_\ell : A^{\otimes \ell} \to \bigoplus_{\lambda=1}^{\ell} A^{\otimes \lambda}$ is def recursively from $\triangleright$ (with 3 different recursions!) ∎

Corollary: $\boxed{\begin{array}{l} \text{Properties of the loop Diff}(A) \text{ for an associative algebra } A: \\ \circ \text{ inversion is unique: } a\backslash 1 = 1/a \\ \circ \text{ right division recovered: } a/b = a \circ (1/b) \\ \quad \text{but not left division: } a\backslash b \neq (a\backslash 1) \circ b \ ! \end{array}}$

Can apply to renormalization: right division as usual while left division new!

12. Other examples of proalgebraic loops or groups

1) Inv(A) extends as a proalgebraic loop to Alg_k (unitary algebras or magmas)
rep. by $H^{inv} = k\{x_n \mid n \geq 1\}$ non-associative pol- (with parenthesizing)
⇝ find $\Delta^{inv}(x_n)$

Let Com_k^*, As_k^*, Alg_k^* be the categories of algebras with (anti-)involution.

2) The algebraic group of unitary elements $\mathcal{U}(A) = \{a \in A \mid a^* a = 1\}$
is represented ~~by~~ on $Com_{\mathbb{R}}^*$ by $H_U = \mathbb{R}[x, x^* \mid xx^* = 1]$
$\Delta(x) = x \otimes x$, $\varepsilon(x) = 1$, $S(x) = x^*$.

- $\mathcal{U}$ ~~and~~ extends to an alg. group on $As_{\mathbb{R}}^*$, rep. by H_U with $\Delta(x) = xy$
$\delta_r(x) = xy^*$, $\delta_\ell(x) = x^* y$.

ex: $\mathcal{U}(M_n(\mathbb{R})) \cong O(n)$ ~~ortog~~ orthogonal gp
$\mathcal{U}(M_n(\mathbb{C})) \cong U(n)$ unitary gp
$\mathcal{U}(M_n(\mathbb{H})) \cong \mathcal{U}(n, \mathbb{H}) \cong Sp(n)$ compact symplectic group

- $\mathcal{U}$ does not extend to $Alg_{\mathbb{R}}^*$, not even as a loop (because $(xy^*)y \neq x$
coconcellation does not hold!)
but it extends to Alternative *-algebras as a loop,
ex: $\mathcal{U}(\mathbb{O}) \cong S^7$, $\mathcal{U}(Zorn(\mathbb{R}))$ not compact ($Zorn(\mathbb{R})$ = matrix Cayley-Dickson algebra)
compare to $\mathcal{U}(\mathbb{H}) \cong S^3$, $\mathcal{U}(\mathbb{P}) \cong SL_2(\mathbb{R})$ groups ($\mathbb{P}$ = split quaternions)

3) Counterexample:
the set of unitary elts in the Cayley-Dickson extension $A \oplus Aj$ with $j^2 = -1$ is
$$\mathcal{U}_{CD}(A) = \{ a + bj \mid (a+bj)^*(a+bj) = 1 \}$$

- it is an algebraic group on $Com_{\mathbb{R}}^*$, rep. by
$$H_{UCD} = \mathbb{R}[x, y, x^*, y^* \mid x^*x + y^*y = 1] \quad \text{with} \quad \Delta(x) = x\otimes x - y \otimes y^* \quad \Delta(y) = x \otimes y + y \otimes x^*$$
$$\varepsilon(x) = 1 \qquad \varepsilon(y) = 0$$
$$S(x) = x^* \qquad S(y) = -y$$

- it does not extend to $As_{\mathbb{R}}^*$ nor as a group nor as a loop,
even if there exist associative non-com. algebras A st $\mathcal{U}_{CD}(A)$ is a loop.
The reason is that A need to be a composition algebra, i.e. an algebra
with a multiplicative quadratic form $N: A \to \mathbb{R}$ st $N(ab) = N(a)N(b)$,
but composition algebras do not have a coproduct $\sqcup$!
(~~here~~ for cocancellation need $x(y^*y) = (y^*y)x$, does not hold in $H \sqcup H$ in $As_{\mathbb{R}}^*$,
unless $y^*y = N(y)$ is scalar.
while holds in $Com_{\mathbb{R}}^*$ where both are $x \otimes x^* x$ in $H \otimes H$!)