Vertex Algebras

Octubre 2019

Outline

1. Vertex algebras
   - Formal Calculus
   - Locality
   - Vertex Algebras
   - Structure of VA

2. Vertex algebras
   - Other equivalent definitions of vertex algebras

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Motivation

All this, was starting even before R. Borcherds introduced the precise mathematical notion of "vertex algebra" in the 1980s

- From String Theory:

  Elementary particles manifest themselves as "vibrational modes" of fundamental strings moving through space, rather than as moving points, according to quantum field-theoretic principles

  A string sweeps out a two-dimensional "world-sheet" in space-time. (Riemann surface).
  The result is two-dimensional conformal (quantum)field theory, formalized in an algebraic spirit (on a physical level of rigor),(A. Belavin,A. Polyakov and A. Zamolodchikov).

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A certain new kind of algebra of operators was emerging, called "operator algebras" based on the "operator product expansion" in quantum field theory.

Roughly speaking, the elements of these "algebras" are certain types of vertex operators.

These are operators of types introduced in the early days of string theory in order to describe certain kinds of physical interactions, at a "vertex," where, for instance, two particles (or strings) enter, and as a result of the interaction, one particle (or string) exits.

These "algebra elements" have certain physically motivated properties.

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Around 1980, most of the finite simple groups-the Chevalley groups and variants- belong to infinite families related to Lie groups.

The largest "sporadic" (not belonging to one of the infinite families) finite simple group, the Fischer-Griess Monster $\mathbb{M}$, had been predicted and was constructed by Griess as a symmetry group (of order about 10^{54}) of a commutative, but very highly nonassociative, seemingly ad hoc new algebra $\mathbb{B}$ of dimension **196883**.

J. Tits proved that $\mathbb{M}$ is actually the full automorphism group of $\mathbb{B}$.

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A bit earlier, J. McKay, J. Thompson, J. Conway and S. Norton had discovered astounding "numerology" culminating in the **"monstrous moonshine" conjectures** concerning the not-yet-proved-to-exist Monster $\mathbb{M}$, namely:

There should exist an infinite-dimensional $\mathbb{Z}$-graded module $V = \oplus_{n\geq -1} V_n$ for $\mathbb{M}$ such that

$$\sum_{n\geq -1} (dim V_n) q^n = J(q),$$

where

$$J(q) = q^{-1} + 0 + 196884q + 21493760q^2 + ...$$

Note: $196884 = 196883 + 1$ - McKay's initial observation. Here J(q) a very well known classical modular function.

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Proving this conjecture would give a remarkable connection between classical number theory and "nonclassical" sporadic group theory.

After Griess constructed $\mathbb{M}$, I. Frenkel, J. Lepowsky and A. Meurman proved the McKayThompson conjecture- that there should exist a natural infinite-dimensional $\mathbb{Z}$-graded $\mathbb{M}$-module V satisfying the condition above.

This was done by means of an explicit (and necessarily elaborate) construction of such a structure V, called the "moonshine module $V^\natural$" The construction heavily uses a network of types of vertex operators and their algebraic structure and relations, yielding the structure $V^\natural$ and a certain "algebra of vertex operators" acting on it, in such a way that the Monster is realized as the automorphism group of this algebra of vertex operators.

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The Monster,finite group, was now understood in terms of a natural infinite-dimensional structure.

Moreover, the 196883-dimensional algebra $\mathbb{B}$ finds itself embedded inside $V^\natural$ in a 196884-dimensional enlargement $\mathcal{B}$ of $\mathbb{B}$, with an identity element adjoined, and this identity element of gives rise to a copy of the Virasoro Lie algebra acting on $V^\natural$.

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Outline

Formal Calculus

We considered formal expressions called **formal distributions** in the indeterminates $z, w, \ldots$ with values in U

$$\sum_{m,n,\ldots \in \mathbb{Z}} a_{m,n,\ldots} z^n w^m \ldots,$$

where $a_{m,n,\ldots}$ are elements of a vector space U over $\mathbb{C}$.
They form a vector space over $\mathbb{C}$ denoted by

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We can always multiply a formal distribution and a Laurent polynomial (provided that product of coefficients is defined), but cannot in general multiply two formal distributions.

Each time when a product of two formal distribution occurs, **we need to check that** it converges in the algebraic sense, i.e. the coefficient of each monomial $z^m w^n$... is a **finite (or convergent) sum.**

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Examples:

(i) $(\sum_{n\geq 0} z^n)(\sum_{n\leq 0} z^n)$

(ii) $(\sum_{n\geq 0} \frac{z^n}{2^n})(\sum_{n\leq 0} z^n)$ (coefficients in this series are convergent, we do not allow this)

(iii) $(\sum_{n\geq 0} z^n)(1-x)(\sum_n z^n) = 0$???????

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Given a formal distribution $a(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, define the **residue** by the usual formula

$$\mathrm{Res}_z a(z) = a_{-1}.$$

Since $\mathrm{Res}_z \partial_z a(z) = 0$, we have the usual *integration by parts formula* (provided that ab is defined):

$$\mathrm{Res}_z \partial_z a(z) b(z) = -\mathrm{Res}_z a(z) \partial_z b(z)$$

Here and further $\partial a(z) = \sum_{n\in\mathbb{Z}} a_n n z^{n-1}$ is the derivative of $a(z)$.

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Let $\mathbb{C}[z, z^{-1}]$ denote the algebra of *Laurent polynomials* in z.

We have a non-degenerate pairing

$$U[[z, z^{-1}]] x \mathbb{C}[z, z^{-1}] \longrightarrow U$$
$$f \times \phi \mapsto, \langle f, \phi \rangle = \text{Res}_z f(z)\phi(z),$$

hence the Laurent polynomials should be viewed as "test functions" for the formal distributions.

Excersice: Formal distributions $a(z)$ and $b(z)$ are equal iff $\langle a(z), \phi(z) \rangle = \langle b(z), \phi(z) \rangle$ for any test function $\phi \in \mathbb{C}[z, z^{-1}]$.

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We introduce the **formal delta function** $\delta(z, w)$ as the following formal distribution in z and w with values in $\mathbb{C}$:

$$\delta(z, w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{w}{z} \right)^n .$$

Notation: Given a rational function $R(z, w)$ with poles only at $z = 0$, $w = 0$ and $|z| = |w|$, we denote by $\iota_{z,w} R$ (resp. $\iota_{z,w} R$) the power series expansion of R in the domain $|z| > |w|$ (resp. $|z| < |w|$).

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For example, we have for $j \in \mathbb{Z}_+$:

$$\iota_{z,w} \frac{1}{(z-w)^{j+1}} = \sum_{m=0}^{\infty} \binom{m}{j} z^{-m-1} w^{m-j},$$

$$\iota_{w,z} \frac{1}{(z-w)^{j+1}} = -\sum_{m=-1}^{-\infty} \binom{m}{j} z^{-m-1} w^{m-j},$$

Thus

$$\partial_w^{(j)} \delta(z,w) = \iota_{z,w} \frac{1}{(z-w)^{j+1}} - \iota_{w,z} \frac{1}{(z-w)^{j+1}} = \sum_{m \in \mathbb{Z}} \binom{m}{j} z^{-m-1} w^{m-j}$$

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Properties of the formal delta function:

a) For any formal distribution $f(z) \in U[[z, z^{-1}]]$ one has:

$$\mathrm{Res}_z f(z)\delta(z,w) = f(w).$$

b) $\delta(z,w) = \delta(w,z)$.

c) $\partial_z\delta(z,w) = -\partial_w\delta(z,w)$.

d) $(z-w)\partial_w^{(j+1)}\delta(z,w) = \partial_w^{(j)}\delta(z,w)$ for $j \in \mathbb{Z}_+$.

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Properties of the formal delta function:

a) For any formal distribution $f(z) \in U[[z, z^{-1}]]$ one has:

$$\mathrm{Res}_z f(z)\delta(z, w) = f(w).$$

b) $\delta(z, w) = \delta(w, z)$.

c) $\partial_z \delta(z, w) = -\partial_w \delta(z, w)$.

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Outline

1. Vertex algebras
   - Formal Calculus
   - Locality
   - Vertex Algebras
   - Structure of VA
2. Vertex algebras
   - Other equivalent definitions of vertex algebras

Question: Characterize the null space of the operator multiplication by $(z-w)^N$ with $N \geq 1$ in $U[[z, z^{-1}, w, w^{-1}]]$.

It is shown in the literature that this space is exactly

$$\sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z,w)\, U[[w, w^{-1}]].$$

Namely, let $N \geq 1$ and $a(z,w) \in U[[z, z^{-1}, w, w^{-1}]]$. Then

$$(z-w)^N a(z,w) = 0$$

if and only if

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We shall often write a formal distribution in the form

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \quad a(z,w) = \sum_{n\in\mathbb{Z}} a_{n,m} z^{-n-1} w^{-m-1}, \text{etc.}$$

This is a natural thing to do since $a_n = \mathrm{Res}_z a(z) z^n$. Then the expansion

$$a(z,w) = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z,w)\, c^j(w)$$

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A formal distribution $a(z, w)$ is called **local** if

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One defines the bracket $[\,,\,]$ on an associative algebra U by letting

$$[a, b] = ab - ba,$$

where a, $b \in U$.

Now, consider a Lie algebra $\mathfrak{g}$.

Two formal distributions $a(z)$ and $b(z)$ with values in a Lie algebra $\mathfrak{g}$ are called **mutually local** (or simply local, or form a local pair) if the formal distribution $[a(z), b(w)] \in \mathfrak{g}[[z, z^{-1}, w, w^{-1}]]$ is local, i.e. if

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Fix a vector space V (the space of states). A formal distribution

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

with values in the ring End V (i.e., $a_n \in \mathrm{End}\,V$) is called a **field** if for any $v \in V$ one has:

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The **normally ordered product** of two fields $a(z)$ and $b(z)$ is defined by

$$: a(z)b(z) := a(z)_+b(z) + b(z)a(z)_- : .$$

This is a field, since given $v \in V$, $b(z)v$ (resp. $a(z)_-v$) is a formal Laurent series (resp. a Laurent polynomial) in z, hence $a(z)_+b(z)v$ (resp. $b(z)a(z)_-v$) is a formal Laurent series in z.

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Given two fiels $a(z)$ and $b(z)$ define the n-th product between fields as:(for $n \in \mathbb{Z}$

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It is easy to check that for $n \in \mathbb{Z}_+$

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Let $glf(V)$ denote the space (over $\mathbb{C}$) of all fields with values in $\mathrm{End}V$.

We have that $glf(V)$ is closed under all the products $a(z)_n b(z)$, $n \in \mathbb{Z}$.

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The **Virasoro algebra** is defined as the Lie algebra $\mathcal{L}$ with basis

$$\{L_n : n \in \mathbb{Z}\} \cap \{c\}$$

equipped with the bracket relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c$$

together with the condition that c is a central element of $\mathcal{L}$.

These relations indeed define a Lie algebra.

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The **Virasoro algebra** is defined as the Lie algebra $\mathcal{L}$ with basis

$$\{L_n : n \in \mathbb{Z}\} \cap \{c\}$$

equipped with the bracket relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c$$

together with the condition that c is a central element of $\mathcal{L}$.

These relations indeed define a Lie algebra.

Consider the formal distribution

$$L(z) = \sum_n L_n z^{-n-2}.$$

The bracket defined above is equivalent to

$$[L(z), L(w)] = \partial_w L(w)\delta(z,w) + 2L(w)\partial_w\delta(z,w) + \frac{C}{12}\partial^3\delta(z,w).$$

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Outline

A **vertex algebra** is a vector space V endowed with a vector $|0\rangle$ (*vacuum vector*),

an endomorphism T (*infinitesimal translation operator*) and linear map of V to the space of fields (*the state-field correspondence*)

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad a_n \in \operatorname{End} V,$$

such that the following axioms hold (a, $b \in V$):

(**translation covariance**): $[T, Y(a,z)] = \partial Y(a,z)$,

(**vacuum**):$T|0\rangle = 0$, $Y(|0\rangle, z) = Id_V$, $Y(a,z)|0\rangle|_{z=0} = a$,

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Remarks:

1) Applying both sides of the translation invariance axiom to $|0\rangle$ we obtain that

$$T(a) = a_{-2}|0\rangle, \qquad (*)$$

from the 1st and 3rd parts of the vacuum axiom after letting $z = 0$.

2) The bracket in translation covariance axiom is the usual bracket: $[T, Y] = TY - YT$, so that this axiom says

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$$a_n|0\rangle = 0 \quad \text{for } n \geq 0, \qquad a_{-1}|0\rangle = a$$

4) Now, applying T to both sides of $(*)$ $n-1$ times, and using $[T, a_n] = -na_{n-i}$ and $T|0\rangle = 0$, we obtain

$$T^{(n)}(a) = a_{-n-1}|0\rangle$$

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Two rather abstract examples.

Example 1)A vertex algebra V is called **holomorphic** if $a_n = 0$ for$n \geq 0$, i.e., $Y(a,z) = \sum_{n \in \mathbb{Z}_+} a_{-n-1} z^n$ formal power series in z.

Let V be a holomorphic vertex algebra. Since the algebra of formal power series in z and w has no zero divisors, it follows that locality for V turns into a usual commutativity:

$$Y(a,z)Y(b,w) = Y(b,w)Y(a,z). \qquad (1)$$

Define a bilinear product ab on the space V by the formula

$$ab = a_{-1}b,$$

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Vertex algebra axioms imply that V is commutative associative unital algebra.

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Furthermore, apply $Y(b, w)$ to both sides of $Y(a, z)|0\rangle = e^{zT}(a)$:

$$Y(b, w)Y(a, z)|0\rangle = Y(b, w)e^{zT}(a).$$

Applying commutativity (locality) to the left-hand side and then $Y(b, w)|0\rangle = e^{wT}(b)$, we obtain

$$Y(a, z)e^{wT}(b) = Y(b, w)e^{zT}(a).$$

Letting $w = 0$ and using the commutativity of our product on V we get

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Finally, translation covariance axiom becomes:

$$T(e^{zT}(a)b) - e^{zT}(a)T(b) = T(e^{zT}(a))b.$$

Letting $z = 0$ we see that T is a derivation of the associative commutative unital superalgebra V.

Conversely, consider V an associative commutative unital algebra V with a derivation T. Let us construct in V a vertex algebra structure: For a, $b \in V$,

define

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This satisfies all axioms of vertex algebras.

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Example 2)

Any local linear field algebra $F \subset glf(U)$ is a vertex algebra with the vacuum vector $|0\rangle = Id_U$, the infintesimal translation operator $T = \partial_z$ and the vertex operators

$$Y(a(z), x)b(z) = \sum_{n \in \mathbb{Z}} (a(z)_n b(z)) x^{-n-1}$$

First, the vertex operators $Y(a(z), x)$ are End$F$-valued fields since F consists of (End$U$-valued) mutually local fields.
Recall that

$$a(z)_n b(z) = \mathrm{Res}_z \left(a(z)b(w)\iota_{z,w}(z-w)^n - b(w)a(z)\iota_{w,z}(z-w)^n \right).$$

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From this expresion and the properties of de δ function, locality axiom follows.

The vacuum axioms mean the following:

$$\partial_z Id_V = 0,\ (Id_V)_n a(z) = \delta_{n,-1} a(z) \text{ for } n \in \mathbb{Z},$$

$$a(z)_n Id_V = \delta_{n,-1} a(z) \text{ for } n \geq -1,$$

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Translation covariance means:

$$[\partial_z, a(z)_n]b(z) = -na(z)_{n-1}b(z) \quad \text{for} n \in \mathbb{Z}.$$

Outline

Skewsymmetry.
For any elements a and b of a vertex algebra V one has the following skewsymmetry relation:

$$Y(a,z)b = e^{zT}Y(b,-z)a.$$

Subalgebras, ideals, and tensor products

A **subalgebra** of a vertex algebra V is a subspace U of V containing $|0\rangle$ such that

$$a_n U \subseteq U \qquad \text{for all } a \in U.$$

It is clear that U is a vertex algebra too, its fields being

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$$a_n U \subseteq U \qquad \text{for all } a \in U.$$

It is clear that U is a vertex algebra too, its fields being

$$Y(a,z) = \sum_n a_n|_U z^{-n-1}.$$

Skewsymmetry.
For any elements a and b of a vertex algebra V one has the following skewsymmetry relation:

$$Y(a,z)b = e^{zT}Y(b,-z)a.$$

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A **homomorphism** of a vertex algebra V to a vertex algebra V' is a linear map $\psi : V \to V'$ such that

$$\psi(a_n b) = \psi(a)_n \psi(b) \quad \text{for all } a, b \in V, \, n \in \mathbb{Z}.$$

An **ideal** of a vertex algebra V is a T-invariant subspace J not containing $|0\rangle$ such that

$$a_n J \subset J \text{ for all } a \in V.$$

Note that we have

$$a_n V \subset J \text{ for all } a \in J.$$

Hence the quotient space V/J has a canonical structure of a vertex algebra, and we have a canonical homomorphism $V \to V/J$ of vertex algebras.

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The **tensor product** of two vertex algebras U and V is defined as follows.

The space of states is $U \otimes V$,

the vacuum vector is $|0\rangle \otimes |0\rangle$,

the infinitesimal translation operator is $T \otimes 1 + 1 \otimes T$,

and fields are

$$Y(u \otimes v, z) = Y(u, z) \otimes Y(v, z) = \sum_{n,m} u_m \otimes v_n z^{-m-n-2}.$$

In other words

$$(u \otimes v)_k = \sum_m u_m \otimes v_{-m+k-1}.$$

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Uniqueness theorem

The following uniqueness theorem is extremely useful in identifying a field with one of the fields of a vertex algebra.

Theorem: Let V he a vertex algebra and let $B(z)$ he a field (with values in $\mathrm{End}\, V$) which is mutually local with all the fields $Y(a,z) \in V$. Suppose that for some $b \in V$:

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The first corollary of the Uniqueness theorem is the following important proposition.

Proposition (n-product axiom) For any two elements a and b of a vertex algebra V and any $n \in \mathbb{Z}$ one has:

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Corollary (a) In a vertex algebra V for any collection of vectors $a^1, ..., a^n \in V$ and any collection of positive integers $j_1 i, ..., j_k$ one has

$$: \partial^{(j_1-1)} Y(a^1, z) .... \partial^{(j_n-1)} Y(a^n, z) := Y(a^1_{-j_1} ... a^n_{-j_n} |0\rangle, z).$$

(b) For any $a, b \in V$ and any $n \in \mathbb{Z}$ one has:

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The following theorem allows one to construct vertex algebras.

Existence Theorem . Let V he a vector superspace, let $|0\rangle \in V$ and T endomorphism of V. Let $\{a^{\alpha}(z)\}_{\alpha \in A}$ (A an index set) be a collection of fields such that
(i) $[T, a^{\alpha}(z)] = \partial a^{\alpha}(z) \quad (\alpha \in A)$,
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(iii) the linear map: $\sum_{\alpha} \mathbb{C} a^{\alpha}(z) \to \sum_{\alpha} \mathbb{C} a^{\alpha}$, defined by $a^{\alpha}(z) \mapsto a^{\alpha}$, is inyective,
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A collection of fields of a vertex algebra V satisfying

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Universal vertex algebras associated to $\mathfrak{g}$

A Lie algebra $\mathfrak{g}$ is called a formal distribution Lie algebra if it is spanned over $\mathbb{C}$ by coefficients of a family F of $\mathfrak{g}$-valued mutually local formal distributions.

Let $\mathfrak{g}$ be a Lie algebra spanned by mutually local formal distributions $a^\alpha(z)$, $(\alpha \in A)$, and suppose that there exists an endomorphism T of the space $\mathfrak{g}$ such that

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$$\mathfrak{g}_{--} = \{a \in \mathfrak{g}\,;\; T^k a = 0 \text{ for } k >> 0\}.$$

Let $\lambda : \mathfrak{g}_{--} \to \mathbb{C}$ be a 1-dimensional $\mathfrak{g}_{--}$-module such that

$$\lambda(T\mathfrak{g}_{--}) = 0.$$

Consider the induced $\mathfrak{g}$-module

$$V^\lambda(\mathfrak{g}) := \mathrm{Ind}_{\mathfrak{g}_{--}}^{\mathfrak{g}} \lambda = U(\mathfrak{g})/U(\mathfrak{g})\langle a - \lambda(a) |\, a \in \mathfrak{g}_{--}\rangle,$$

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Note that the formal distributions $a^{\alpha}(z)$ are represented in $V^{\lambda}(\mathfrak{g})$ by fields (which we shall denote by the same symbol).

The derivation T of $\mathfrak{g}$ extends to a derivation of $U(\mathfrak{g})$, which can be pushed down to an endomorphism of the space $V^{\lambda}(\mathfrak{g})$ since $\lambda(T\mathfrak{g}_{--}) = 0$. This endomorphism is again denoted by T.

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The following theorem is now an immediate corollary of the Existence.

Theorem. Let $\mathfrak{g}$ be a regular formal distribution Lie algebra. Then the $\mathfrak{g}$-module $V^\lambda(\mathfrak{g})$ has a unique vertex algebra structure with $|0\rangle$ the vacuum vector and generated by the fields $a^\alpha(z)$ $(\alpha \in A)$.

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Outline

- This is due to B. Bakalov and V. Kac. (Field Algebras)

Let $(V, |0\rangle, T)$ be a vector space with a distinguished vector and endomorphism, and let Y be a **state field correspondance**, namely it linear map $a \mapsto Y(a,z)$ of a vector space V to the space of End$V$ -valued fields satisfying

(translation invariance) $[T, Y(a,z)] = Y(Ta,z) = \partial_z Y(a,z)$.

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For $a, b \in V$, we define their λ-**product** by the formula

$$a_\lambda b = \mathrm{Res}_z e^{\lambda z} Y(a,z)b = \sum_{n \geq 0,\, finite} a_n b(\lambda)^{(n)}.$$

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$$|0\rangle.a = a = a.|0\rangle,$$

while the translation invariance axiom shows that:

$$T(a.b) = (Ta).b + a.(Tb)$$

and

$$T(a_\lambda b) = (Ta)_\lambda b + a_\lambda(Tb),$$
$$(Ta)_\lambda b = -\lambda a_\lambda b \text{ for all } a, b \in V.$$

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Conversely, if we are given a linear operator T, a λ-product and a .-product on V, satisfying the above properties, we can reconstruct the state field correspondence Y by the formulas:

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On the other hand, with respect to the .-product, V is a $(\mathbb{C}[T]-)$ differential algebra (i.e., an algebra with a derivation T) with a unit $|0\rangle$.

A $\mathbb{C}[T]$-module V, equipped with a linear map $V \otimes V \to \mathbb{C}[\lambda] \otimes V$, $a \otimes b \to a\lambda b$, satisfying

$$T(a_\lambda b) = (Ta)_\lambda b + a_\lambda (Tb),$$

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Summarizing:

Giving a state field correspondence on a vector space V with a distinguished vector $|0\rangle$ is equivalent to providing V with a structure of a $\mathbb{C}[T]$-conformal algebra and a structure of a $\mathbb{C}[T]$-differential algebra with a unit $|0\rangle$.

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To have a vertex algebra we still have to deal with locality!!!!

An algebra satisfying

$$a.(b.c) - b.(a.c) = (a.b - b.a).c$$

for all a, b, $c \in V$ is called **left-symmetric.**

A **Liebnitz conformal algebra** is a $\mathbb{C}[T]$-conformal algebra such that the following Jacobi identity holds:

$$(a_\lambda b)_{\lambda+\mu} c = a_\lambda(b_\mu c) - b_\mu(a_\lambda c).$$

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Theorem Giving a vertex algebra structure on a vector space V with a distinguished vector $|0\rangle$ is the same as providing V with the structures of a Lie $\mathbb{C}[T]$-conformal algebra and a left symmetric $\mathbb{C}[T]$-differential algebra with a unit $|0\rangle$, satisfying the Wick formula and

$$a.b - b.a = \int_{-T}^{0} a_\lambda b \, d\lambda \quad a, \, b \in V.$$

• Lepowsky-Li's definition of vertex algebra.

Notation:

Delta function:

$$\delta(x) = \sum_{n\in\mathbb{Z}} \in \mathbb{C}[[x, x^{-1}]].$$

Binomial expansion: For $n \in \mathbb{Z}$

$$(x+y)^n = \sum_{j\in\mathbb{Z}_+} \binom{n}{j} x^{n-j} y^j$$

where $\binom{n}{j} = \frac{n(n-1)(n-2)....(n-j+1)}{j!}$

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A **vertex algebra** consists of a vector space V together with a distinguished element $\mathbf{1} \in V$ called *vacuum* vector and a linear map

$$Y(\cdot, x) : V \to (\mathrm{End}\ V)[[x, x^{-1}]]$$
$$u \mapsto Y(u, x) = \sum_{n \in \mathbb{Z}} u_n x^{-n-1},$$

satisfying the following axioms:

- **Truncation**: For all $u, v \in U$,
$$Y(u, x)v \in V((x)). \tag{1}$$
This axiom is equivalent to requiring $u_n v = 0$ for $n >> 0$.
- **Left unit**: For all $u \in V$,
$$Y(\mathbf{1}, \mathbf{x})\mathbf{u} = \mathbf{u}. \tag{2}$$
- **Creation:** For all $u \in V$, $Y(u, x)\mathbf{1}$ is a holomorphic power series in x and
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- **Jacobi identity:** For all $u, v \in V$,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1)$$

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Remarks:

• They don't ask T to be part of the definition! They show that if we define $T(v) = v_{-2}\mathbf{1}$ for all $v \in V$ then

$$Y(Tv, x) = \frac{d}{dx} Y(v, x).$$

They show that skew symmetry

$$Y(u, x)v = e^{zT} Y(v, -x)u$$

also holds starting from Jacobi identity instead of locality.

Using skew symmetry they show that

$$[T, Y(v, x)] = \frac{d}{dx} Y(v, x) = Y(Tv, x).$$

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They deduce Locality from Jacobi identity.

Then the prove the following

Proposition: The Jacobi identity for a vertex algebra follows from weak commutativity in the presence of the other axioms together with the T-bracket-derivative formula . In particular, in the definition of the notion of vertex algebra, the Jacobi identity can be replaced by these properties.

Namely, both definitions are equivalent

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