Vertex Algebras

Octubre 2019
Outline

1. Vertex algebras
   - Formal Calculus
   - Locality
   - Vertex Algebras
   - Structure of VA

2. Vertex algebras
   - Other equivalent definitions of vertex algebras
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   - Formal Calculus
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   - Other equivalent definitions of vertex algebras
Motivation

All this, was starting even before R. Borcherds introduced the precise mathematical notion of "vertex algebra" in the 1980s

- From String Theory:

  Elementary particles manifest themselves as "vibrational modes" of fundamental strings moving through space, rather than as moving points, according to quantum field-theoretic principles.

  A string sweeps out a two-dimensional "world-sheet" in space-time. (Riemann surface).

  The result is two-dimensional conformal (quantum)field theory, formalized in an algebraic spirit (on a physical level of rigor), (A. Belavin, A. Polyakov and A. Zamolodchikov).
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A certain new kind of algebra of operators was emerging, called "operator algebras" based on the "operator product expansion" in quantum field theory.

Roughly speaking, the elements of these "algebras" are certain types of vertex operators.

These are operators of types introduced in the early days of string theory in order to describe certain kinds of physical interactions, at a "vertex," where, for instance, two particles (or strings) enter, and as a result of the interaction, one particle (or string) exits.

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From Mathematics: "Monstrous Moonshine" Program to classify finite simple groups.

Around 1980, most of the finite simple groups—the Chevalley groups and variants—belong to infinite families related to Lie groups.

The largest "sporadic" (not belonging to one of the infinite families) finite simple group, the Fischer-Griess Monster $M$, had been predicted and was constructed by Griess as a symmetry group (of order about $10^{54}$) of a commutative, but very highly nonassociative, seemingly ad hoc new algebra $B$ of dimension $196883$.

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J. Tits proved that $\mathbb{M}$ is actually the full automorphism group of $\mathbb{B}$. 
A bit earlier, J. McKay, J. Thompson, J. Conway and S. Norton had discovered astounding "numerology" culminating in the *monstrous moonshine* conjectures concerning the not-yet-proved-to-exist Monster $\mathbb{M}$, namely:

There should exist an infinite-dimensional $\mathbb{Z}$-graded module $V = \oplus_{n \geq -1} V_n$ for $\mathbb{M}$ such that

$$\sum_{n \geq -1} (\dim V_n)q^n = J(q),$$

where

$$J(q) = q^{-1} + 0 + 196884q + 21493760q^2 + \ldots$$

Note: $196884 = 196883 + 1$ - McKay's initial observation. Here $J(q)$ a very well known classical modular function.
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Proving this conjecture would give a remarkable connection between classical number theory and "nonclassical" sporadic group theory.

After Griess constructed $\mathbb{M}$, I. Frenkel, J. Lepowsky and A. Meurman proved the McKay-Thompson conjecture— that there should exist a natural infinite-dimensional $\mathbb{Z}$-graded $\mathbb{M}$-module $V$ satisfying the condition above.

This was done by means of an explicit (and necessarily elaborate) construction of such a structure $V$, called the "moonshine module $V^\natural$". The construction heavily uses a network of types of vertex operators and their algebraic structure and relations, yielding the structure $V^\natural$ and a certain "algebra of vertex operators" acting on it, in such a way that the Monster is realized as the automorphism group of this algebra of vertex operators.
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The Monster, finite group, was now understood in terms of a natural infinite-dimensional structure.

Moreover, the 196883-dimensional algebra $B$ finds itself embedded inside $V^\natural$ in a 196884-dimensional enlargement $\mathcal{B}$ of $B$, with an identity element adjoined, and this identity element of gives rise to a copy of the Virasoro Lie algebra acting on $V^\natural$. 
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   - Other equivalent definitions of vertex algebras
We considered formal expressions called **formal distributions** in the indeterminates $z, w, \ldots$ with values in $U$

$$\sum_{m,n,\ldots \in \mathbb{Z}} a_{m,n,\ldots} z^n w^m \ldots,$$

where $a_{m,n,\ldots}$ are elements of a vector space $U$ over $\mathbb{C}$. They form a vector space over $\mathbb{C}$ denoted by

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$$U[[z, z^{-1}, w, w^{-1}, ...]].$$
We can always multiply a formal distribution and a Laurent polynomial (provided that product of coefficients is defined), but cannot in general multiply two formal distributions.

Each time when a product of two formal distribution occurs, we need to check that it converges in the algebraic sense, i.e. the coefficient of each monomial $z^m w^n ...$ is a finite (or convergent) sum.
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Examples:
(i) \((\sum_{n\geq 0} z^n)(\sum_{n\leq 0} z^n)\)

(ii) \((\sum_{n\geq 0} \frac{z^n}{2^n})(\sum_{n\leq 0} z^n)\) (coefficients in this series are convergent, we do not allow this)

(iii) \((\sum_{n\geq 0} z^n)(1-x)(\sum_n z^n) = 0 \)
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Given a formal distribution \( a(z) = \sum_{n \in \mathbb{Z}} a_n z^n \), define the residue by the usual formula

\[
\text{Res}_z a(z) = a_{-1}.
\]

Since \( \text{Res}_z \partial_z a(z) = 0 \), we have the usual integration by parts formula (provided that \( ab \) is defined):

\[
\text{Res}_z \partial_z a(z) b(z) = -\text{Res}_z a(z) \partial_z b(z)
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Here and further \( \partial a(z) = \sum_{n \in \mathbb{Z}} a_n n z^{n-1} \) is the derivative of \( a(z) \).
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Let $\mathbb{C}[z, z^{-1}]$ denote the algebra of *Laurent polynomials* in $z$.

We have a non-degenerate pairing

$$U[[z, z^{-1}]] \times \mathbb{C}[z, z^{-1}] \rightarrow U$$

$$f \times \phi \mapsto \langle f, \phi \rangle = \text{Res}_z f(z)\phi(z),$$

hence the Laurent polynomials should be viewed as "test functions" for the formal distributions.

**Exercise:** Formal distributions $a(z)$ and $b(z)$ are equal iff

$$\langle a(z), \phi(z) \rangle = \langle b(z), \phi(z) \rangle$$

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We introduce the **formal delta function** \( \delta(z, w) \) as the following formal distribution in \( z \) and \( w \) with values in \( \mathbb{C} \):

\[
\delta(z, w) = z^{-1} \sum_{n \in \mathbb{Z}} \left( \frac{w}{z} \right)^n.
\]

**Notation:** Given a rational function \( R(z, w) \) with poles only at \( z = 0 \), \( w = 0 \) and \( |z| = |w| \), we denote by \( \iota_{z,w}R \) (resp. \( \iota_{z,w}R \)) the power series expansion of \( R \) in the domain \( |z| > |w| \) (resp. \( |z| < |w| \)).
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For example, we have for $j \in \mathbb{Z}_+$:

$$
\frac{1}{(z - w)^{j+1}} = \sum_{m=0}^{\infty} \binom{m}{j} z^{-m-1} w^{m-j},
$$

$$
\frac{1}{(z - w)^{j+1}} = -\sum_{m=-1}^{-\infty} \binom{m}{j} z^{-m-1} w^{m-j},
$$

Thus

$$
\partial^{(j)}_{w} \delta(z, w) = \frac{1}{(z - w)^{j+1}} - \frac{1}{(z - w)^{j+1}} = \sum_{m \in \mathbb{Z}} \binom{m}{j} z^{-m-1} w^{m-j}
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For example, we have for \( j \in \mathbb{Z}_+ \):

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Properties of the formal delta function:

a) For any formal distribution \( f(z) \in U[[z, z^{-1}]] \) one has:

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\text{Res}_z f(z) \delta(z, w) = f(w).
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b) \( \delta(z, w) = \delta(w, z) \).

c) \( \partial_z \delta(z, w) = -\partial_w \delta(z, w) \).

d) \( (z - w) \partial_w^{(j+1)} \delta(z, w) = \partial_w^{(j)} \delta(z, w) \) for \( j \in \mathbb{Z}_+ \).

e) \( (z - w)^{j+1} \partial_w^{(j)} \delta(z, w) = 0 \) for \( j \in \mathbb{Z}_+ \)
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\text{Res}_z f(z) \delta(z, w) = f(w).
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d) \( (z - w) \partial_w^{(j+1)} \delta(z, w) = \partial_w^{(j)} \delta(z, w) \) for \( j \in \mathbb{Z}_+ \).

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1. Vertex algebras
   - Formal Calculus
   - Locality
   - Vertex Algebras
   - Structure of VA

2. Vertex algebras
   - Other equivalent definitions of vertex algebras
Question: Characterize the null space of the operator multiplication by \((z - w)^N\) with \(N \geq 1\) in \(U[[z, z^{-1}, w, w^{-1}]]\).

It is shown in the literature that this space is exactly

\[
\sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z, w) U[[w, w^{-1}]].
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Namely, let \(N \geq 1\) and \(a(z, w) \in U[[z, z^{-1}, w, w^{-1}]]\). Then

\[(z - w)^N a(z, w) = 0\]

if and only if

\[
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We shall often write a formal distribution in the form

\[ a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a(z, w) = \sum_{n \in \mathbb{Z}} a_{n,m} z^{-n-1} w^{-m-1}, \text{ etc.} \]

This is a natural thing to do since \( a_n = \text{Res}_z a(z) z^n \). Then the expansion

\[ a(z, w) = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z, w) c^j(w) \]

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A formal distribution \( a(z, w) \) is called **local** if

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This expansion is called the **OPE expansion** of \( a(z, w) \) and the \( c^j(w) \) are called the **OPE coefficients** of \( a(z, w) \).
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One defines the bracket $[ , ]$ on an associative algebra $U$ by letting

$$[a, b] = ab - ba,$$

where $a, b \in U$.

Now, consider a Lie algebra $\mathfrak{g}$.

Two formal distributions $a(z)$ and $b(z)$ with values in a Lie algebra $\mathfrak{g}$ are called **mutually local** (or simply local, or form a local pair) if the formal distribution $[a(z), b(w)] \in \mathfrak{g}[[z, z^{-1}, w, w^{-1}]]$ is local, i.e. if

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Differentiating both sides by $z$ and multiplying by $(z - w)$, we see that the locality of $a(z)$ and $b(z)$ implies the locality of $\partial a(z)$ and $b(z)$.
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Fix a vector space $V$ (the space of states). A formal distribution

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

with values in the ring $\text{End} \ V$ (i.e., $a_n \in \text{End} \ V$) is called a field if for any $v \in V$ one has:

$$a_n v = 0 \quad \text{for } n \gg 0.$$

This means that $a(z)v$ is a formal Laurent series in $z$ (i.e., $a(z)v \in V[[z]][z^{-1}]$).
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The **normally ordered product** of two fields $a(z)$ and $b(z)$ is defined by

$$: a(z)b(z) := a(z) + b(z) + b(z)a(z) : .$$

This is a field, since given $v \in V$, $b(z)v$ (resp. $a(z)v$) is a formal Laurent series (resp. a Laurent polynomial) in $z$, hence $a(z) + b(z)v$ (resp. $b(z)a(z)v$) is a formal Laurent series in $z$.

Thus, the space of fields forms an algebra with respect to the normally ordered product (which is in general neither commutative nor associative).
The **normally ordered product** of two fields $a(z)$ and $b(z)$ is defined by

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The derivative $\partial a(z)$ of a field $a(z)$ is again a field and $\partial$ is a derivation of the normally ordered product:

$$\partial : a(z)b(z) := \partial a(z)b(z) + a(z)\partial b(z),$$

since $(\partial a(z))_\pm = \partial(a(z)_\pm)$.

Given two fields $a(z)$ and $b(z)$ define the $n$-th product between fields as:

$$(for \ n \in \mathbb{Z})$$

$$a(z)_nb(z) = \text{Res}_z (a(z)b(w)_{z,w}(z-w)^n - b(w)a(z)_{w,z}(z-w)^n).$$

It is easy to check that for $n \in \mathbb{Z}_+$

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**Dong’s Lemma:** If $a(z)$, $b(z)$ and $c(z)$ are pairwise mutually local fields (resp. formal distributions), then $a(z)_n b(z)$ and $c(z)$ are mutually local fields (resp. formal distributions) for all $n \in \mathbb{Z}$ (resp. $n \in \mathbb{Z}_+$). In particular: $a(z) b(z)$ and $c(z)$ are mutually local fields provided that $a(z)$, $b(z)$ and $c(z)$ are.

Let $glf(V)$ denote the space (over $\mathbb{C}$) of all fields with values in $\text{End} V$.

We have that $glf(V)$ is closed under all the products $a(z)_n b(z)$, $n \in \mathbb{Z}$.

This is called the **general linear field algebra**.
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A subspace $F$ of $glf(V)$ containing the identity operator $ld_V$, and closed under all the products $n$-th products (then automatically $\partial_z F \subset F$) is called a **linear field algebra**.

A linear field algebra is called **local** if it consists of mutually local fields.

A subspace $F$ of $glf(V)$ is a linear field algebra iff $ld_V \in F$, $\partial F \subset F$, $F$ is closed under normally ordered product $F$ is closed under OPE (i.e., all the OPE) are in $F$).
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Computing an OPE

The Virasoro algebra is defined as the Lie algebra $\mathcal{L}$ with basis

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equipped with the bracket relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c$$

together with the condition that $c$ is a central element of $\mathcal{L}$.

These relations indeed define a Lie algebra.
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\[ L(z) = \sum_n L_n z^{-n-2}. \]

The bracket defined above is equivalent to

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Outline

1. Vertex algebras
   - Formal Calculus
   - Locality
   - Vertex Algebras
   - Structure of VA

2. Vertex algebras
   - Other equivalent definitions of vertex algebras
A **vertex algebra** is a vector space $V$ endowed with a vector $|0\rangle$ (*vacuum vector*), an endomorphism $T$ (*infinitesimal translation operator*) and linear map of $V$ to the space of fields (*the state-field correspondence*)

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a_n \in \text{End} V,$$

such that the following axioms hold ($a, b \in V$):

**translation covariance**: $[T, Y(a, z)] = \partial Y(a, z),$

**vacuum**: $T|0\rangle = 0, \quad Y(|0\rangle, z) = \text{Id}_V, \quad Y(a, z)|0\rangle|_{z=0} = a,$

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Vertex Algebras
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Remarks:

1) Applying both sides of the translation invariance axiom to $|0\rangle$ we obtain that

$$T(a) = a_{-2}|0\rangle,$$

from the 1st and 3rd parts of the vacuum axiom after letting $z = 0$.

2) The bracket in translation covariance axiom is the usual bracket: $[T, Y] = TY - YT$, so that this axiom says

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3) The first of the vacuum axioms says that $|0\rangle_n = \delta_{n,-1}$ and the second part says that

$$a_n |0\rangle = 0 \quad \text{for } n \geq 0, \quad a_{-1} |0\rangle = a$$

4) Now, applying $T$ to both sides of $(\ast)$ $n - 1$ times, and using $[T, a_n] = -n a_{n-1}$ and $T|0\rangle = 0$, we obtain

$$T^{(n)}(a) = a_{-n-1} |0\rangle$$

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Two rather abstract examples.

Example 1) A vertex algebra $V$ is called **holomorphic** if $a_n = 0$ for $n \geq 0$, i.e., $Y(a, z) = \sum_{n \in \mathbb{Z}^+} a_{-n-1} z^n$ formal power series in $z$.

Let $V$ be a holomorphic vertex algebra. Since the algebra of formal power series in $z$ and $w$ has no zero divisors, it follows that locality for $V$ turns into a usual commutativity:

$$Y(a, z) Y(b, w) = Y(b, w) Y(a, z). \quad (1)$$

Define a bilinear product $ab$ on the space $V$ by the formula

$$ab = a_{-1} b,$$

and let $|0\rangle = 1$

Vertex algebra axioms imply that $V$ is commutative associative unital algebra.
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Vertex algebra axioms imply that $V$ is commutative associative unital algebra.
Furthermore, apply \( Y(b, w) \) to both sides of \( Y(a, z)|0\rangle = e^{zT}(a) \):

\[
Y(b, w)Y(a, z)|0\rangle = Y(b, w)e^{zT}(a).
\]

Applying commutativity (locality) to the left-hand side and then \( Y(b, w)|0\rangle = e^{wT}(b) \), we obtain

\[
Y(a, z)e^{wT}(b) = Y(b, w)e^{zT}(a).
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Letting \( w = 0 \) and using the commutativity of our product on \( V \) we get

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Thus, the fields \( Y(a, z) \) are defined entirely in terms of the product on \( V \) and the operator \( T \).
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Thus, the fields $Y(a, z)$ are defined entirely in terms of the product on $V$ and the operator $T$. 
Finally, translation covariance axiom becomes:

$$T(e^{zT}(a)b) - e^{zT}(a)T(b) = T(e^{zT}(a))b.$$ 

Letting $z = 0$ we see that $T$ is a derivation of the associative commutative unital superalgebra $V$.

Conversely, consider $V$ an associative commutative unital algebra $V$ with a derivation $T$. Let us construct in $V$ a vertex algebra structure: For $a, b \in V$,

define

$$Y(a, z)b = (e^{zT}(a))b.$$ 

This satisfies all axioms of vertex algebras.
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If $T = 0$, then $Y(a, z)(b) = ab$. Therefore we may view vertex algebras as a generalization of unital commutative associative algebras where the multiplication depends on the parameter $z$ via

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However, as we shall see, a general vertex algebra is very far from being a "commutative" object.
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Example 2)

Any local linear field algebra $F \subset glf(U)$ is a vertex algebra with the vacuum vector $\langle 0 \rangle = Id_U$, the infinitesimal translation operator $T = \partial_z$ and the vertex operators

$$Y(a(z), x)b(z) = \sum_{n \in \mathbb{Z}} (a(z)_n b(z)) x^{-n-1}$$

First, the vertex operators $Y(a(z), x)$ are $\text{End}F$-valued fields since $F$ consists of ( $\text{End}U$-valued) mutually local fields. Recall that

$$a(z)_n b(z) = \text{Res}_z (a(z)b(w)_{z,w}(z - w)^n - b(w)a(z)_{w,z}(z - w)^n).$$
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Thus

\[ Y(a(z), x)b(z) = \sum_{n \in \mathbb{Z}} (a(z)_n b(z)) x^{-n-1} \]

\[ = \text{Res}_z (a(z)b(w) \nu_{z,w} \delta(z - w, x)) \]

\[ - b(w)a(z) \nu_{w,z} \delta(z - w, x)) \].

From this expression and the properties of the \( \delta \) function, locality axiom follows.

The vacuum axioms mean the following:

\[ \partial_z ld_V = 0, \quad (ld_V)_n a(z) = \delta_{n,-1} a(z) \text{ for } n \in \mathbb{Z}, \]

\[ a(z)_n ld_V = \delta_{n,-1} a(z) \text{ for } n \geq -1, \]
Thus

\[ Y(a(z), x)b(z) = \sum_{n \in \mathbb{Z}} (a(z)_n b(z)) x^{-n-1} \]

\[ = \operatorname{Res}_z (a(z) b(w) \iota_{z,w} \delta(z - w, x) - b(w) a(z) \iota_{w,z} \delta(z - w, x)) . \]

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Translation covariance means:

\[ [\partial_z, a(z)_n] b(z) = -na(z)_{n-1} b(z) \quad \text{for} \quad n \in \mathbb{Z}. \]
1. **Vertex algebras**
   - Formal Calculus
   - Locality
   - Vertex Algebras
   - Structure of VA

2. **Vertex algebras**
   - Other equivalent definitions of vertex algebras
Skewsymmetry. For any elements $a$ and $b$ of a vertex algebra $V$ one has the following skewsymmetry relation:

$$Y(a, z)b = e^{zT}Y(b, -z)a.$$ 

Subalgebras, ideals, and tensor products

A **subalgebra** of a vertex algebra $V$ is a subspace $U$ of $V$ containing $|0\rangle$ such that

$$a_n U \subseteq U$$

for all $a \in U$.

It is clear that $U$ is a vertex algebra too, its fields being

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A **homomorphism** of a vertex algebra $V$ to a vertex algebra $V'$ is a linear map $\psi : V \to V'$ such that

$$\psi(a_n b) = \psi(a)_n \psi(b) \quad \text{for all } a, b \in V, n \in \mathbb{Z}.$$ 

An **ideal** of a vertex algebra $V$ is a $T$-invariant subspace $J$ not containing $|0\rangle$ such that

$$a_n J \subseteq J \quad \text{for all } a \in V.$$ 

Note that we have

$$a_n V \subseteq J \quad \text{for all } a \in J.$$ 

Hence the quotient space $V/J$ has a canonical structure of a vertex algebra, and we have a canonical homomorphism $V \to V/J$ of vertex algebras.
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The space of states is $U \otimes V$,

the vacuum vector is $|0\rangle \otimes |0\rangle$,

the infinitesimal translation operator is $T \otimes 1 + 1 \otimes T$,

and fields are

$$Y(u \otimes v, z) = Y(u, z) \otimes Y(v, z) = \sum_{n,m} u_m \otimes v_n z^{-m-n-2}.$$ 

In other words

$$(u \otimes v)_k = \sum_m u_m \otimes v_{-m+k-1}.$$ 

It is straightforward to check that $U \otimes V$ is a vertex algebra.
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It is straightforward to check that $U \otimes V$ is a vertex algebra.
Uniqueness theorem

The following uniqueness theorem is extremely useful in identifying a field with one of the fields of a vertex algebra.

**Theorem:** Let $V$ be a vertex algebra and let $B(z)$ be a field (with values in $\text{End} V$) which is mutually local with all the fields $Y(a, z) \in V$. Suppose that for some $b \in V$:

$$B(z)|0\rangle = e^{zT}b.$$

Then $B(z) = Y(b, z)$. 
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\[
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Then \( B(z) = Y(b, z) \).
The first corollary of the Uniqueness theorem is the following important proposition.

**Proposition** ($n$-product axiom) For any two elements $a$ and $b$ of a vertex algebra $V$ and any $n \in \mathbb{Z}$ one has:

$$Y(a_nb, z) = Y(a, z)_n Y(b, z).$$
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**Corollary** (a) In a vertex algebra $V$ for any collection of vectors $a^1, \ldots, a^n \in V$ and any collection of positive integers $j_1, \ldots, j_k$ one has

$$
: \partial^{(j_1-1)} Y(a^1, z) \ldots \partial^{(j_n-1)} Y(a^n, z) : = Y(a_{-j_1} \ldots a_{-j_n} |0\rangle, z).
$$

(b) For any $a, b \in V$ and any $n \in \mathbb{Z}$ one has:

$$
: \partial^{(n)} Y(a, z) Y(b, z) : = Y(a_{-n-1} b, z).
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(c) For any $a \in V$ one has $Y(Ta, z) = \partial Y(a, z)$. 
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The following theorem allows one to construct vertex algebras.

**Existence Theorem.** Let $V$ be a vector superspace, let $|0\rangle \in V$ and $T$ endomorphism of $V$. Let $\{a^\alpha(z)\}_{\alpha \in A}$ ($A$ an index set) be a collection of fields such that

(i) $[T, a^\alpha(z)] = \partial a^\alpha(z)$ ($\alpha \in A$),
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Then the formula

$$Y(a_{j_1}^{\alpha_1} \ldots a_{j_n}^{\alpha_n}|0\rangle, z) = a_1^{\alpha_1}(z)_{j_1}(a_2^{\alpha_2}(z)_{j_2} \ldots (a_n^{\alpha_n}(z)_{j_n}Id_V))$$

defines a unique structure of a vertex algebra on $V$ such that $|0\rangle$ is the vacuum vector, $T$ is the infinitesimal translation operator and

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A collection of fields of a vertex algebra $V$ satisfying

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Universal vertex algebras associated to $g$

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**Universal vertex algebras associated to $\mathfrak{g}$**

A Lie algebra $\mathfrak{g}$ is called a formal distribution Lie algebra if it is spanned over $\mathbb{C}$ by coefficients of a family $F$ of $\mathfrak{g}$-valued mutually local formal distributions.

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Then $\mathfrak{g}$ is called a **regular** formal distribution Lie algebra.

It is clear that $T$ is a derivation of the Lie algebra $\mathfrak{g}$ given by the formula

$$Ta^\alpha_n = -na^\alpha_{n-1}.$$
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Let
\[ g_- = \{ a \in g ; T^k a = 0 \text{ for } k \gg 0 \} \].

Let \( \lambda : g_- \to \mathbb{C} \) be a 1-dimensional \( g_- \)-module such that
\[ \lambda(Tg_-) = 0. \]

Consider the induced \( g \)-module
\[ V^\lambda(g) := \text{Ind}_{g_-}^g \lambda = U(g)/U(g) \langle a - \lambda(a) | a \in g_- \rangle, \]
and let \( |0\rangle \in V^\lambda(g) \) be the image of \( 1 \in U(g) \).
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Note that the formal distributions $a^\alpha(z)$ are represented in $V^\lambda(g)$ by fields (which we shall denote by the same symbol).

The derivation $T$ of $g$ extends to a derivation of $U(g)$, which can be pushed down to an endomorphism of the space $V^\lambda(g)$ since $\lambda(Tg\ldots) = 0$. This endomorphism is again denoted by $T$. 
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The following theorem is now an immediate corollary of the Existence.

**Theorem.** Let $g$ be a regular formal distribution Lie algebra. Then the $g$-module $V^\lambda(g)$ has a unique vertex algebra structure with $|0\rangle$ the vacuum vector and generated by the fields $a^\alpha(z)$ ($\alpha \in A$).

The vertex algebras $V^\lambda(g)$ are called **universal vertex algebras associated to** $g$. 
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1. Vertex algebras
   - Formal Calculus
   - Locality
   - Vertex Algebras
   - Structure of VA

2. Vertex algebras
   - Other equivalent definitions of vertex algebras
This is due to B. Bakalov and V. Kac. (Field Algebras)

Let \((V, |0\rangle, T)\) be a vector space with a distinguished vector and endomorphism, and let \(Y\) be a \textbf{state field correspondance}, namely it linear map \(a \mapsto Y(a, z)\) of a vector space \(V\) to the space of \(\text{End} V\)-valued fields satisfying

(translation invariance) \([T, Y(a, z)] = Y(Ta, z) = \partial_z Y(a, z)\).

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\(Y(|0\rangle, z) = ld_V\) and \(Y(a, z)|0\rangle = a + T(a)z + \cdots \in V[[z]]\).
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For $a, b \in V$, we define their $\lambda$-product by the formula

$$a_\lambda b = \text{Res}_z e^{\lambda z} Y(a, z)b = \sum_{n \geq 0, \text{finite}} a_n b(\lambda)^{(n)}.$$

We also have the $(-1)$-st product on $V$, which we denote as

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a_{\lambda}b = \text{Res}_z e^{\lambda z} Y(a, z)b = \sum_{n \geq 0, \text{finite}} a_n b(\lambda)^{(n)}.
\]

We also have the \((-1)\)-st product on \( V \), which we denote as

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while the translation invariance axiom shows that:

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Conversely, if we are given a linear operator $T$, a $\lambda$-product and a $\cdot$-product on $V$, satisfying the above properties, we can reconstruct the state field correspondence $Y$ by the formulas:

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A $\mathbb{C}[T]$-module $V$, equipped with a linear map $V \otimes V \to \mathbb{C}[\lambda] \otimes V$, $a \otimes b \to a\lambda b$, satisfying

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is called a ($\mathbb{C}[T]$-)conformal algebra.

On the other hand, with respect to the $\cdot$-product, $V$ is a ($\mathbb{C}[T]$-)differential algebra (i.e., an algebra with a derivation $T$) with a unit $|0\rangle$. 
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Summarizing:

Giving a state field correspondence on a vector space $V$ with a distinguished vector $|0\rangle$ is equivalent to providing $V$ with a structure of a $\mathbb{C}[T]$-conformal algebra and a structure of a $\mathbb{C}[T]$-differential algebra with a unit $|0\rangle$.

To have a vertex algebra we still have to deal with locality!!!!
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To have a vertex algebra we still have to deal with locality!!!!
An algebra satisfying

\[ a.(b.c) - b.(a.c) = (a.b - b.a).c \]

for all \( a, b, c \in V \) is called \textbf{left-symmetric}.

A \textbf{Liebnitz conformal algebra} is a \( \mathbb{C}[T] \)-conformal algebra such that the following Jacobi identity holds:

\[ (a_\lambda b)_{\lambda+\mu} c = a_\lambda (b_\mu c) - b_\mu (a_\lambda c). \]

And the \textbf{Wick formula} is

\[ a_\lambda (b.c) = (a_\lambda b).c + b.(a_\lambda c) + \int_0^\lambda (a_\lambda b)_\mu c d\mu. \]
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Theorem: Giving a vertex algebra structure on a vector space $V$ with a distinguished vector $|0\rangle$ is the same as providing $V$ with the structures of a Lie $\mathbb{C}[T]$-conformal algebra and a left symmetric $\mathbb{C}[T]$-differential algebra with a unit $|0\rangle$, satisfying the Wick formula and

$$a.b - b.a = \int_{-T}^{0} a_{\lambda} b d\lambda \quad a, b \in V.$$
Lepowsky-Li’s definition of vertex algebra.

**Notation:**

Delta function:

\[ \delta(x) = \sum_{n \in \mathbb{Z}} \in \mathbb{C}[[x, x^{-1}]]. \]

Binomial expansion: For \( n \in \mathbb{Z} \)

\[ (x + y)^n = \sum_{j \in \mathbb{Z}_+} \binom{n}{j} x^{n-j} y^j \]

where \( \binom{n}{j} = \frac{n(n-1)(n-2)\ldots(n-j+1)}{j!} \)
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where \( \left( \begin{array}{c} n \\ j \end{array} \right) = \frac{n(n-1)(n-2)\ldots(n-j+1)}{j!} \)
A **vertex algebra** consists of a vector space $V$ together with a distinguished element $1 \in V$ called *vacuum* vector and a linear map 

$$Y(\cdot, x) : V \rightarrow (\text{End } V)[[x, x^{-1}]]$$

$$u \mapsto Y(u, x) = \sum_{n \in \mathbb{Z}} u_n x^{-n-1},$$

satisfying the following axioms:

- **Truncation**: For all $u, v \in V$,

  $$Y(u, x)v \in V((x)).$$

  This axiom is equivalent to requiring $u_n v = 0$ for $n >> 0$.

- **Left unit**: For all $u \in V$,

  $$Y(1, x)u = u.$$

- **Creation**: For all $u \in V$, $Y(u, x)1$ is a holomorphic power series in $x$ and

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Jacobi identity: For all \( u, v \in V \),

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \ Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \ Y(v, x_2) Y(u, x_1)

= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \ Y(Y(u, x_0)v, x_2).
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Remarks:

- They don’t ask $T$ to be part of the definition! They show that if we define $T(v) = v_{-2}$ for all $v \in V$ then

$$Y(Tv, x) = \frac{d}{dx} Y(v, x).$$

They show that skew symmetry

$$Y(u, x)v = e^{x T} Y(v, -x)u$$

also holds starting from Jacobi identity instead of locality.

Using skew symmetry they show that

$$[T, Y(v, x)] = \frac{d}{dx} Y(v, x) = Y(Tv, x).$$

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• Locality is called **weak commutativity**.

They deduce Locality from Jacobi identity.

Then they prove the following

**Proposition:** The Jacobi identity for a vertex algebra follows from weak commutativity in the presence of the other axioms together with the \( T \)-bracket-derivative formula. In particular, in the definition of the notion of vertex algebra, the Jacobi identity can be replaced by these properties.

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